THE TWENTY-FOURTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fourth annual meeting of the Society was held in New York City on Thursday and Friday, December 27–28, 1917, extending through two sessions on Thursday and a morning session on Friday. The attendance included the following forty-six members:

President R. J. Aley, Professor Joseph Bowden, Dr. T. H. Brown, Professor W. B. Carver, Professor F. N. Cole, Dr. G. M. Conwell, Dr. J. V. DePorte, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Dr. T. R. Hollcroft, Professor L. A. Howland, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor P. H. Linehan, Professor C. R. MacInnes, Professor Helen A. Merrill, Professor R. L. Moore, Professor Frank Morley, Dr. G. W. Mullins, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Clara E. Smith, Professor D. E. Smith, Professor Sarah E. Smith, Professor H. D. Thompson, Mr. J. N. Vedder, Professor C. W. Watkeys, Mr. H. E. Webb, Mr. R. A. Wetzel, Professor E. E. Whitford, Professor Ruth G. Wood, Professor J. W. Young.

At the opening session Professor R. G. D. Richardson took the chair. Professor J. W. Young presided at the following sessions. The Council announced the election of the following persons to membership in the Society: Dr. J. W. Campbell, Wesley College, Winnipeg, Canada; Dr. Mary F. Curtis, Western Reserve University; Mr. C. H. Parsons, Columbia University; Mr. J. B. Rosenbach, University of New Mexico; Mr. H. M. Terrill, Columbia University. Five applications for membership in the Society were received.

Committees were appointed to arrange for the summer meeting at Dartmouth College in 1918 and for the summer meeting and colloquium at the University of Chicago in 1919.

The total membership of the Society is now 735, including 77 life members. The total attendance of members at all

meetings, including sectional meetings, during the past year was 351; the number of papers read was 167. The number of members attending at least one meeting during the year was 217. At the annual election 116 votes were cast. The Treasurer's report shows a balance of $9,762.98, including the life-membership fund of $6,333.13. Sales of the Society's publications during the year amounted to $1,474.19. The Library now contains 5,475 volumes, excluding unbound dissertations.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professor J. L. COOLIDGE,
Professor D. R. CURTISS.

Secretary, Professor F. N. COLE.

Treasurer, Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1920,

Professor R. C. ARCHIBALD,
Professor DUNHAM JACKSON,
Professor D. N. LEHMER,
Professor J. B. SHAW.

The following papers were read at the annual meeting:

(1) Professor F. L. HITCHCOCK: "The coincident points of two algebraic transformations."

(2) Professor W. B. CARVER: "The conditions for the failure of the Clifford chain."

(3) Professor C. J. KEYSER: "The rôle of the concept of infinity in the work of Lucretius."

(4) Professor C. J. KEYSER: "Concerning the number of possible interpretations of any system of postulates."

(5) Dr. W. H. WILSON: "Systems of functional equations which define hyperbolic sine, hyperbolic cosine, sine, and cosine uniquely."

(6) Dr. C. H. FORSYTH: "Tangential interpolation of ordinates among areas."

(7) Professor W. B. FITE: "Concerning the zeros of the solutions of certain differential equations."

(8) Professor R. L. MOORE: "Concerning a set of postulates for plane analysis situs."

(9) Dr. C. A. FISCHER: "Integral equations involving Stieltjes integrals."

(10) Professor O. E. GLENN: "Preliminary report on a new treatment of theorems of finiteness."

(11) Professor C. L. E. MOORE: "Rotations in hyperspace."

(12) Dr. G. M. GREEN: "Memoir on the general theory of surfaces and rectilinear congruences."

(13) Dr. J. F. RITT: "On the iteration of rational functions."

(14) Dr. OLIVE C. HAZLETT: "On rational integral invariants and covariants of the general linear algebra."

(15) Professor ANNA J. PELL: "Systems of linear equations."

(16) Dr. NORBERT WIENER: "Internal isomorphisms of complex algebra."

(17) Dr. T. R. HOLLCROFT: "A classification of general (2, 3) point correspondences between two planes."

(18) Professor W. F. OSGOOD: "Singular points of analytic transformations."

(19) Dr. M. T. HU: "Linear integro-differential equations with a boundary condition."

(20) Professor FRANK MORLEY: "Some general projective invariants of the algebraic planar curve."

The papers of Professor Hitchcock and Dr. Wilson were communicated to the Society through Professor C. L. E. Moore. Dr. Wiener was introduced by Professor Huntington. Dr. Hu's paper was communicated through Professor Bôcher. In the absence of the authors the papers of Professor Hitchcock, Dr. Wilson, Dr. Forsyth, Professor Glenn, Professor C. L. E. Moore, Professor Osgood, and Dr. Hu were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Let $x_1, x_2, \cdots, x_n$ be a set of n independent variables. For convenience of language we may say that these variables are the homogeneous coordinates of a point in space of $n - 1$

dimensions. Let $P_1, P_2, \cdots, P_n$ and $Q_1, Q_2, \cdots, Q_n$ be two sets of polynomials homogeneous in the variables x and of degrees p and q respectively, defining a pair of transformed points P and Q. There always exist one or more points x such that P and Q coincide. This is the same as saying that the two-row determinants of the matrix

$$P_1, P_2, \cdots, P_n$$

$$Q_1, Q_2, \cdots, Q_n$$

vanish simultaneously for certain properly chosen sets of values of the variables $x_1, x_2, \cdots, x_n$. Professor Hitchcock shows that the number of such points is in general $p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + q^{n-1}$.

2. Clifford discovered a chainwise extension of the notion of the circumcircle of 3 lines, and showed* that n lines determine a circle when n is odd and a point when n is even. Professor Morley has given† analytic expressions for the Clifford circles and points in conjugate complex coordinates, and has shown that the Clifford circle becomes a straight line when a certain determinant vanishes. In the present paper Professor Carver shows that the Clifford chain fails when, and only when, the n lines are tangents to certain metric curves. These curves are rational, and of a type the dual (projectively) of the Jonquière curves.

3. Professor Keyser shows that the scheme of thought in the De Rerum Natura of Lucretius is dominated by the concept of infinity; that the author's conception of an infinite multitude accords with the current definition of the term; that the same is true of his notion of an infinite magnitude; that the infinites of Lucretius are not variables but are, like those of Cantor, static affairs; that, on the other hand, the Lucretian infinites are not composed of abstract things (points of space and pure numbers) but of concrete things (material particles, for example); that a Lucretian infinite multitude is always of the denumerable variety; and that the poet made repeated use of the characteristic whole-part property of infinite multitudes.

* "Synthetic proof of Miquel's theorem," Clifford's Mathematical Papers, p. 38 (1870).

† "On the metric geometry of the plane n-line," *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 97.

4. The main purpose of this note by Professor Keyser is to show that any postulate system involving at least one undefined term denoting an element as distinguished from a relation admits of any given infinite number of different interpretations.

5. In his Lehrbuch der Funktionentheorie, second edition, page 582, Professor Osgood has given a determination of the functions $\sin x$ and $\cos x$ on the basis of their addition theorems. In the present paper Dr. Wilson determines the functions $f(x) = \sin ax$ and $g(x) = \cos ax$ as the general continuous solutions of the simultaneous equations

$$g(x - y) = g(x)g(y) + f(x)f(y),$$
$$f(x - y) = f(x)g(y) - g(x)f(y),$$

where x and y are independent real variables and $f(x) \not\equiv 0$. It is furthermore shown that if x and y are not real and if $x = u + v\sqrt{-1}$, u and v real, then the general solutions continuous over the finite complex x-plane are

$$g(x) = \cos(au + bv), \qquad f(x) = \sin(au + bv),$$

the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of reals are

$$g(x) = \cos bv, \qquad f(x) = \sin bv,$$

and the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of imaginaries are

$$g(x) = \cos au, \qquad f(x) = \sin au,$$

where a and b are arbitrary constants. A similar determination of the functions $g(x) = \cosh ax$ and $f(x) = \sinh ax$ is made from the simultaneous equations

$$g(x - y) = g(x)g(y) - f(x)f(y),$$
$$f(x - y) = f(x)g(y) - g(x)f(y).$$

6. Dr. Forsyth modifies an interpolation formula used to interpolate ordinates when the data consist of areas so that in interpolating several ordinates in each of several successive intervals the discontinuities ordinarily occurring at the points

of intersection of any two consecutive interpolation curves are practically eliminated by requiring the slopes of these curves to be the same at the intersections.

7. In this paper Professor Fite generalizes the results of Kneser published in volume 42 of the *Mathematische Annalen* concerning certain linear differential equations whose solutions change sign an infinite number of times as the independent variable increases indefinitely through real values. He also establishes the existence of a minimum length for an interval within which all the functions which form a solution of a system of linear equations can vanish.

8. In his paper "On the foundations of plane analysis situs,"* Professor Moore exhibited three systems of axioms, Σ_1, Σ_2, and Σ_3. He proposes to show that every space that satisfies Σ_1 or Σ_2 is a number plane in the sense that it is in one-to-one continuous correspondence with euclidean space of two dimensions.

9. In this paper Dr. Fischer extends the definition of the Stieltjes integral $\int f(x)d\alpha(x)$ so that it is determined uniquely whenever both f and α have finite variation. It is then proved that when $f(x)$ and $K(x, y)$ have finite variation, and when the variation of $K(x, y)$ considered as a function of y alone is bounded, the equation

$$f(x) = u(x) + \lambda \int_a^b u(y)d_yK(x, y)$$

has a unique, finite solution, for sufficiently small values of λ. It is also proved that this equation can under certain conditions be put into the form

$$g(x) = u(x) + \lambda \int_a^b L(x, y)du(y),$$

and that this last equation has a solution for sufficiently small values of λ, when it is merely assumed that $g(x)$ has finite variation, that $L(x, y)$ is finite, and that when it is considered as a function of x alone its variation is bounded. Both the method of successive substitutions and that of reciprocal functions are employed.

* *Transactions*, vol. 17, no. 2 (April, 1916), p. 131.

10. The rationale of the method of Professor Glenn's paper is based upon a property, which he has established, that all concomitants of any one of several designated infinite systems are expressible as polynomials in certain elemental concomitants belonging to specified domains. Hilbert's theorem then applies.

The theory is adaptable to formal concomitants of a quantic to a prime modulus, concerning which further investigation is in progress.

11. Professor Moore first shows that a complex 2-vector in a space of $2m$ dimensions can always be resolved into the sum of m mutually completely perpendicular plane vectors. This resolution is not unique in case certain relations exist between products of these vectors. In four dimensions the resolution is not unique if the 2-vector is self-complementary. From these facts it is shown that a rotation in $2m$ dimensions can be resolved into rotations parallel to m mutually completely perpendicular planes. This resolution is not unique if the rates of rotation in two or more of the planes are equal.

12. In Dr. Green's paper certain geometric concepts introduced by him in previous communications to the Society are applied in the study of a curved surface and of certain associated rectilinear congruences. A line l in the tangent plane of the surface has related to it a line l' passing through the corresponding point of the surface, the lines l and l' being reciprocal polars with respect to the osculating quadric of that point. The lines l and l' are in the general relation R with respect to the asymptotic net of the surface, according to a definition introduced by Dr. Green in other papers.* A congruence Γ composed of lines l determines a reciprocal congruence Γ' of lines l'. By further specializing the relation between the congruences Γ and Γ' various important congruences may be defined which are uniquely determined by the surface. Such specialization of the relation between Γ and Γ' seems to be very fruitful both in the discovery of new congruences and in the characterization of known ones. Among these are the directrix congruences of Wilczynski, a pair of reciprocal congruences which reduce to Sullivan's

*Cf. in particular a recent note in the *Proceedings of the National Academy of Sciences*, vol. 3, no. 10 (Oct., 1917).

scroll directrix congruences when the surface is ruled, and the canonical congruences and pseudo-normal congruence introduced by Dr. Green at the summer meeting of 1916 and described in the note already cited.

A congruence Γ is said to be harmonic to a surface S if its developables correspond to a conjugate net on S, and a congruence Γ' is said to be conjugate to S if its developables correspond to a conjugate net on S. Characterizations are given for such congruences, and various questions are studied in connection with them. In particular, the characterization of planar nets with equal invariants is found to be closely related to the case in which the developables of the congruence Γ' are indeterminate.

The developables of a congruence Γ correspond to a net of curves on S, and the tangents to these curves are called Γ-tangents. The line l is met in its focal points by the tangents which are conjugate to the corresponding Γ-tangents, a property which has some important geometric consequences, in particular the geometric characterization of congruences harmonic to a surface.

13. Let $f_n(x)$ be the nth iterate of a rational function $f(x)$. The theorems presented by Dr. Ritt deal principally with the antecedents of any point a, that is, the points x such that, for some n, $f_n(x) = a$, and with the associates of a, the points x such that, for some n, $f_n(x) = f_n(a)$. The antecedents of a point are generally infinite in number, and the set of their limiting points contains a perfect set, which probably is an analytic curve; a similar statement holds for the associates. Perhaps the most interesting result given, part of which was found before by Professor A. A. Bennett, and which is an immediate consequence of the existence of certain meromorphic functions discovered by Poincaré, is the following:

If $f(a) = a$, and $|f'(a)| > 1$, then, given any point b, there is an antecedent of b in any neighborhood of a, however small.

It is shown also that in any neighborhood of such a point a there are an infinite number of points x such that $f_n(x) = x$ for some n.

14. Dr. Hazlett's paper proves some theorems on rational integral invariants and covariants for general linear algebras, analogous to well-known theorems in the classical invariant

theory for algebraic forms. For instance, such invariants must be homogeneous, and must possess a certain property which (by reason of the analogy with the classical invariant theory for algebraic forms) is called isobarism. Similarly for covariants. By the aid of these and other fundamental properties, it follows at once that the rational integral invariants are finite. In the cases $n = 1, 2$ all algebraic invariants and covariants are invariants and covariants respectively of the characteristic determinants. This is the converse of a theorem proved in a previous paper.

15. By means of an orthogonal system of linear forms and linear differential forms, Mrs. Pell shows that the theory of the system of linear equations

$$x_i = c_i + \sum_k a_{ik}x_k + \sum_\alpha \int_a^b \frac{da_i^{(\alpha)}(t)du^{(\alpha)}(t)}{da_0^{(\alpha)}(t)},$$

$$u^{(\alpha)}(s) = f(s) + \sum_k a_k^{(\alpha)}(s)x_k + \sum_\beta \int_a^b \frac{dA^{(\alpha,\beta)}(s,t)du^{(\beta)}(t)}{da_0^{(\beta)}(t)},$$

where $\{x_i\}$ and $\{u^{(\alpha)}(s)\}$ are unknown, is reduced to the theory of the linear equations

$$x_i = d_i + \sum_k b_{ik}x_k.$$

A special case is the theory of mixed linear integral equations. Existence and properties of solutions of a symmetric system are obtained.

16. An internal isomorphism of a system of elements and operations thereon is a one-to-one correspondence between the elements of the system and themselves which transforms operations belonging to the system into operations belonging to the system. Two such isomorphisms are considered as the same if they induce the same transformation of the operations of the system, independently of what they do to the elements. Dr. Wiener shows that the internal isomorphisms of the system in which the elements are the complex numbers and the operations are such that a polynomial in the operations and the operands with integral coefficients is zero, are all established by linear transformations.

17. A general (2, 3) point correspondence between two

planes (x) and (x') is said to exist when to any point of (x') correspond three points of (x) and to any point of (x), two points of (x') such that the correspondence in neither plane is involutorial. Such a correspondence is defined by pairs of algebraic equations in the coordinates (x_1, x_2, x_3) and (x_1', x_2', x_3') whose loci are systems of curves in the planes (x) and (x') intersecting in three and two points respectively. Each such system gives rise to a transformation relating curves of one plane to curves of the other and determines in each plane a locus of points two of whose image points in the other plane coincide, and likewise the loci of these coincident image points. The system is said to be independent if it cannot be reduced to another by any series of birational transformations. In all, Dr. Hollcroft finds twelve such independent systems and finally proves that any system defining a general (2, 3) point correspondence between two planes is birationally equivalent to some one of these twelve types.

18. A preliminary notice of Professor Osgood's paper appeared in the BULLETIN for June, 1917. The paper will appear in full in the *Transactions*.

19. Dr. Hu studies a linear integro-differential equation of a very general type with a linear boundary condition, and introduces such conceptions as adjoint systems and Green's functions.

20. After a suggestion for obtaining the discriminant, Professor Morley considers the degree of the conditions that a line meet the curve thrice at one point and twice at another, or twice at three points.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE fortieth meeting of the Chicago Section was held at the University of Chicago on December 28–29 in conjunction with the third annual meeting of the Mathematical Association of America. The meeting opened with a dinner of the two organizations on Thursday evening, December 27, at which seventy-three persons were present. The program for the joint session on Friday afternoon consisted of the retiring address by the chairman of the section, Professor W. B. Ford, entitled "A conspectus of the modern theory of divergent series," and of a paper by Professor L. D. Ames, "On a definition of the real number system by means of infinite decimals."

The attendance at the different sessions, which continued until Saturday afternoon, included about eighty-five persons, among whom were the following fifty-nine members of the Society:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. A. Bliss, Professor P. P. Boyd, Professor H. T. Burgess, Professor W. H. Bussey, Professor W. D. Cairns, Professor Florian Cajori, Professor D. F. Campbell, Professor R. D. Carmichael, Professor C. E. Comstock, Professor A. R. Crathorne, Mr. G. H. Cresse, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Professor E. L. Dodd, Professor Arnold Dresden, Professor Otto Dunkel, Professor Arnold Emch, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor W. B. Ford, Professor Harris Hancock, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor A. M. Kenyon, Dr. H. R. Kingston, Professor Kurt Laves, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor G. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor C. C. Morris, Professor E. J. Moulton, Professor G. W. Myers, Professor H. B. Phillips, Professor L. C. Plant, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Dr. G. W. Smith, Professor E. J. Townsend, Professor H. W. Tyler,

Professor J. N. Van der Vries, Professor L. G. Weld, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. A. Young.

At the business meeting on Saturday forenoon the following officers were elected: chairman, Professor G. A. Bliss; secretary, Professor Arnold Dresden; third member of programme committee, Dr. A. J. Kempner. The section decided to elect its officers for terms of two years, the new practice to start with the officers elected at this meeting. A committee was appointed to consider the advisability of holding a symposium at the next spring meeting and to make the necessary arrangements in case the decision is favorable.

The sessions of Friday forenoon and Saturday forenoon were presided over by Professor Ford, chairman of the Section; the joint session of Friday afternoon by the president of the Society, Professor L. E. Dickson, and the session of Saturday afternoon by Professor G. A. Bliss, chairman-elect of the Section.

The following papers were presented at this meeting:

(1) Dr. G. E. WAHLIN: "A study of the principal units of an algebraic domain $\mathfrak{k}(\mathfrak{p}, \mathfrak{a})$."

(2) Dr. C. H. FORSYTH: "Supernormal curves."

(3) Professor E. L. DODD: "A comparison of certain functions of measurement, assuming the Gaussian probability law which provides for a constant error."

(4) Dr. E. W. CHITTENDEN: "On the limit functions of sequences of continuous functions converging relatively uniformly."

(5) Dr. A. R. SCHWEITZER: "On the iterative functional compositions of index $(n+1, k)$, $(n, k = 1, 2, \cdots)$ and associated functional equations."

(6) Dr. A. R. SCHWEITZER: "On a description of mathematics."

(7) Professor ARNOLD EMCH: "On the theory of isotropic curves."

(8) Professor H. B. PHILLIPS: "The relation of cubical determinants to the double multiplication of Gibbs."

(9) Professors H. B. PHILLIPS and C. L. E. MOORE: "The dyadics occurring in a projective point space of three dimensions."

(10) Professor R. D. CARMICHAEL: "On sequences of integers defined by recurrence relations."

(11) Professor C. N. MOORE: "On the convergence of the developments in Bessel's functions."

(12) Professor G. A. MILLER: "Sets of independent generators of a substitution group."

(13) Professor J. B. SHAW: "Infinite linear associative algebras."

(14) Professor H. T. BURGESS: "Note on the reduction of a pencil of singular quadratic forms."

(15) Dr. H. R. KINGSTON: "On isothermal nets of curves."

(16) Dr. A. J. KEMPNER: "On polynomials and their residue systems."

(17) Dr. A. L. NELSON: "Note on seminvariants of systems of partial differential equations."

(18) Professor G. A. BLISS: "Solutions of differential equations as functions of the constants of integration."

(19) Professor H. J. ETTLINGER: "On the non-self-adjoint linear system of the second order."

(20) Professor R. D. CARMICHAEL: "Repeated solutions of a certain class of linear functional equations."

The papers of Drs. Wahlin, Forsyth, Chittenden, Kingston, Kempner, and Nelson were read by title. Abstracts, numbered to correspond to the titles in the list above, follow below.

1. In a recent number of the BULLETIN Dr. Wahlin published an article in which he gave a study of the group of residues of the principal units of $\mathfrak{f}(\mathfrak{p}, \mathfrak{a})$ with respect to a certain modulus.

The present paper is a closer study of the principal units along the same lines. The principal units are classified into regular and irregular units, the classification being dependent upon a certain congruence property. Then the smallest number of irregular units that can occur in the base of the group of residues is determined.

2. Perhaps the most useful curve is the Gaussian or probability curve whose equation is $y = y_0 e^{-(x^2/k)}$ or, written in the complete form,

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2},$$

where N is the total area of the curve and σ is the standard deviation, usually defined as the square root of the second

unit moment about the mean. However, the curve is ill adapted for one of its most common uses—that of graduation, because the equation is too rigid, having essentially only one parameter. Dr. Forsyth proposes in this paper what amounts to the addition of a second parameter by allowing the exponent of x to be any even integer, the general equation becoming $y = y_0 e^{-x^{2n}/k}$. The values of y_0 and k are then expressed in terms of moments which may be computed for any distribution proposed for graduation. A formula is then derived for determining the most appropriate value of n to be used for graduating a given distribution. A short table of ordinates of the curve for $n = 2$ is added to be used in graduations.

3. Professor Dodd makes a comparison of functions of measurements, postulating that the probability that the error of a measurement will be x is

$$\frac{h}{\sqrt{\pi}} e^{-h^2(x-c)^2}\, dx,$$

where c is the unknown constant error. A comparison with c not zero cannot always be directly inferred from the corresponding comparison with c equal to zero, as was shown previously.* The following extension is, however, immediate: The arithmetic mean is superior to any other weighted mean and to the median, from the standpoint of error-risk, even with c not zero. But if the constant error c is present, then the more numerous the measurements, the more nearly certain it is that the arithmetic mean will incorporate this error c. Thus, in certain cases where an actual "hitting of the mark" is required, the making of measurements beyond a certain number is prejudicial to accuracy; although in the usual case—where "good average results" are desired—the more numerous the measurements, the more satisfactory will be the arithmetic mean, even when a constant error is present.

4. According to Professor E. H. Moore a sequence of functions f_n defined on a range $\mathfrak{P}$ converges relatively uniformly to a limit function f if there exists a scale function σ defined on $\mathfrak{P}$ such that for every small positive number e there exists an integer n_e such that for all $n \geqq n_e$

$$|f_n - f| \leqq e\,|\sigma|$$

uniformly on $\mathfrak{P}$.

* *Monatshefte für Mathematik und Physik*, vol. 24 (1913), pp. 270, 271.

If the functions f_n are continuous on an interval $\mathfrak{P} \equiv (a, b)$, it is shown by Dr. Chittenden that the discontinuities of the limit function f relative to any perfect subset $\mathfrak{P}_0$ of $\mathfrak{P}$ cannot be dense on $\mathfrak{P}_0$ if the convergence is relatively uniform. The condition is also sufficient. From this result it is easy to show that the convergence of Fourier series is not always relatively uniform. Various consequences of this result in relation to the Baire system of classification of functions are considered.

5. Dr. Schweitzer constructs* the complete ordered set of iterative compositions of order k and degree $n + 1$ ($n, k = 1, 2, 3, \cdots$), i. e., iterative compositions of k functions each of which contains not more than $n + 1$ variables. Two cases are distinguished according as the k functions all have the same number of variables or not. In the former case one obtains the complete list of compositions of order $k > 1$ and homogeneous degree, merely by assigning suffixes 1, 2, 3, $\cdots$ in all possible ways to the functional symbols involved in the set of compositions when $k = 1$ and correspondingly affecting by superscripts the arguments of the new functions introduced. In the latter case the required iterative compositions are obtained from the compositions of the former case by increasing or diminishing in all possible ways the number of variables involved, the operation being subject to the notation of the functional symbols of which the variables are arguments.

In the second part of the paper the author constructs "systematically" functional equations corresponding to the preceding iterative compositions by means of the iteration, inversion, elimination, or transformation of variables. A general equation in iterative compositions with all its variables notationally distinct is defined, homogeneous or non-homogeneous with reference to order and degree. Solutions of simultaneous systems are obtained. Examples:

I. $\phi_1\{\phi(x_1, \cdots, x_{n+1}), t_1, \cdots, t_n\} = \phi\{\phi_1(x_1, t_1, \cdots, t_n), \cdots, \phi_1(x_{n+1}, t_1, \cdots, t_n)\}$. For $n = 1$ solutions of this functional equation have been given by Hankel† and the author.

II. $f\{\phi(x, z), f(x, y)\} = \psi(y, z)$, $f_1\{f(x, z), f(x, y)\} = \psi(y, z)$, $f_2\{\phi(x, z), f(x, y)\} = f(y, z)$.

* Cf. BULLETIN, vol. 23, pp. 301, 302. In Theorem III on p. 302 the specific group of order four is 1, (14)(23), (14), (23).

† Theorie der Complexen Zahlen, p. 34.

III. $\phi\{\phi_1(x_{11}, \cdots, x_{1k}), \cdots, \phi_1(x_{n1}, \cdots, x_{nk})\} = \phi_1\{\phi(x_{11}, \cdots, x_{n1}), \cdots, \phi(x_{1k}, \cdots, x_{nk})\}$ $(k \leqq n, n = 2, 3, \cdots)$.

IV. $f_1\{f_1(x_{11}, \cdots, x_{1n}), \cdots, f_1(x_{n1}, \cdots, x_{nn})\} = f\{f_1(x_{11}', \cdots, x_{1n}'), \cdots, f_1(x_{n1}', \cdots, x_{nn}')\}$, where x_{ik}' are certain linear homogeneous transformations of x_{ik} with coefficients m_{ik}. In connection with equations of this type it is shown that if

$$f(x_{11}, \cdots, x_{1n}) = \sum_{i=1}^{n} a_{1i}x_{1i}$$

and

$$f_1(x_{11}, \cdots, x_{1n}) = \sum_{i=1}^{n} a_{1i}'x_{1i},$$

the coefficients a_{1i}' are linear homogeneous functions of the coefficients a_{1i} with m's as coefficients. Consequently a group of iterative functional equations of the second order is readily defined.

6. Dr. Schweitzer's definition of mathematics is as follows: mathematics is the systematic construction of clear concepts of real numerical reference, mediate or immediate. The author carefully outlines his meaning, philosophically, of the words in this definition; in particular, his interpretation of "systematic" is very largely Neo-Kantian.

7. Professor Emch's paper deals with the generation of rational circular, in particular, with rational isotropic curves. Every rational curve in a complex z'-plane may be obtained as the transformation of the real axis of a complex z-plane by a rational transformation $z' = f(z)/g(z)$, with coefficients from the domain of ordinary complex (including real) numbers. By such a transformation an algebraic curve is transformed into a curve of the same deficiency, so that rational curves are transformed into rational curves. Necessary and sufficient conditions are established for the polynomials $f(z)$ and $g(z)$, in order that the transformed curves may be circular and, in particular, isotropic. It is shown how these conditions depend upon the theorem: if a polynomial equation $g(z) = 0$, with real coefficients, has imaginary roots only, then $g(z)$ may be represented as the sum of the squares of two polynomials of the same type.

As an application, the generation of all bicircular rational

quartics is given explicitly. For example, Bernoulli's lemniscate

$$(x'^2 + y'^2)^2 - 2a^2(x'^2 - y'^2) = 0$$

is represented by

$$z' = \frac{-a\sqrt{2}(z^2 - 1)}{z^2 + 2iz + 1}.$$

8. In this paper, Professor Phillips shows that a cubical determinant of the nth order is the Gibbs multiple product of n dyadics in a space of n dimensions. More generally, the multiple product of m (less than n) such dyadics is a cubical matrix of the nth order with m sheets. As a particular application, it is shown that these cubical matrices can be used instead of determinants in Hedrick's law of analogy which states that a theorem in cubical determinants corresponds to each homogeneous identity in ordinary algebra.

9. If we classify dyadics by the dimensions of their factors, there are four types of dyadics in three dimensions: the one-three and three-one, representing collineations; the one-one and three-three, representing correlations; the two-two, not in general representing a contact transformation; and the one-two, two-three, and their conjugates. Using the symbolic notation introduced by Phillips in a former paper, Professors Phillips and Moore give a treatment of the foregoing types of dyadics and the invariants obtained from them by simple and double multiplication with each other and with idemfactors.

This paper will appear in the *Proceedings of the American Academy of Arts and Sciences.*

10. Special classes of recurrent sequences of integers have been investigated in detail by Lucas and others and numerous interesting theorems have resulted from such study. The purpose of the present paper by Professor Carmichael is to develop certain general properties of the general sequence of integers $u_0, u_1, u_2, \cdots$ defined uniquely in terms of the given initial numbers $u_0, u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = \alpha,$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. The leading fundamental properties of such sequences are developed by

elementary means and several applications of them are given. In particular, many theorems due to Fermat (among them some of his most remarkable discoveries) are readily derived from general results to which one is thus led naturally by elementary means. This association of so many results of Fermat with a single elementary theory tends to suggest the conjecture that the methods of this paper may have been employed by Fermat in some of his investigations.

11. The principal result of Professor Moore's paper is contained in the following theorem: If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, and in an interval $0 \leqq x \leqq c < 1$ has a first derivative which has a Lebesgue integral and which approaches a finite limit as x approaches $+0$, then the development of $f(x)$ in Bessel's functions of order $\nu(\nu \geqq 0)$ will be convergent to the value $f(+0)$ for $x = 0$, provided $\nu = 0$ or $f(+0) = 0$, and will be uniformly convergent to the value $f(x)$ in the interval $0 \leqq x \leqq c$, provided $\nu = 0$ or $f(+0) = 0$.

12. A set of λ substitutions contained in a given substitution group G is called a set of independent generators of G whenever these λ substitutions generate G, but no $\lambda - 1$ of them have this property. It is clearly always possible to select more than one set of independent generators of G whenever the order of G exceeds 2, and λ is not always the same for the various possible sets of generators of G. In particular, when G is the symmetric group of degree n it is well known that sets can be so chosen that λ may have any value from 2 to $n - 1$, but it does not seem to have been proved that λ can never exceed $n - 1$ when G is a substitution group of degree n. When G is cyclic it is easily seen that λ may have any value from one to the number of different prime factors of the order of G, but it can have no other value.

In the present paper Professor Miller proves the following theorems: If G is any substitution group of degree n with k systems of intransitivity it is always possible to generate G by $n - k$ of its substitutions. When n is written in the form

$$a_0 p^m + a_1 p^{m-1} + \cdots + a_m,$$

where m is a positive integer and each of the positive integers $a_0, a_1, \cdots, a_m$ is less than the prime number p, then the

number of substitutions in each of the possible sets of independent generators in the Sylow subgroup of order p^a contained in G is $a_0 m + a_1(m-1) + \cdots + a_{m-1}$. If one of the transitive constituents of the intransitive subgroup of index p contained in a transitive group G of order p^s is abelian, the number of the independent generators of G cannot exceed one plus the number of the independent generators of this transitive constituent increased by the number of its independent generators whose order exceeds p.

13. In Professor Shaw's paper the general theorems of linear associative algebra are extended from algebras of a finite number of units to algebras with an infinite number of units. These algebras are of two kinds: one having an infinite number of idempotents, the other without idempotents and having no characteristic equations. The latter are called infinitipotent algebras. The most general linear associative algebra will then consist of one infinite or finite semi-simple subalgebra, an infinite or finite nilpotent subalgebra, and an infinitipotent subalgebra. The general theorem of finite algebras is extended to infinite algebras. Some applications are made to recent analysis along the lines of integral equations and integro-differential equations.

14. The reduction of a pencil of singular quadratic forms whose matrix $\lambda A + B$ is singular for all values of λ has been accomplished in a variety of ways. None of the schemes found in the literature enable one to compute the matrix of the reducing substitution with any degree of satisfaction.

Professor Burgess outlines in this note a simple and direct method of accomplishing the reduction, and at the same time computing the matrix of the reducing substitution.

15. In a former paper read before the Society Dr. Kingston discussed some metric properties of nets of plane curves by means of a completely integrable system of three linear partial differential equations of the second order. In particular, the condition for isothermality of the nets was found. In the present paper he exhibits the forms assumed by these equations, when the nets are the various isothermal systems of conics.

16. The following theorem is known: Given any integers

$\alpha_1, \cdots, \alpha_p, 0 \leqq \alpha_\mu < p$, p prime, then there exist polynomials with integral coefficients such that the α form a complete residue system modulo p of the polynomial. If we demand that $f(\mu) \equiv \alpha_\mu \bmod p$ and that $f(x)$ be of degree less than p, the polynomial is uniquely determined modulo p.*

The results obtained by Dr. Kempner may be stated as follows: Let $m = p_1^{\beta_1} \cdots p_\mu^{\beta_\mu}$, and $\gamma_1, \cdots, \gamma_m$ any integers satisfying $0 \leqq \gamma < m$, then there is usually no polynomial $f(x)$ with integral coefficients for which $\gamma_1, \cdots, \gamma_m$ form a complete residue system modulo m. Necessary and sufficient relations between the γ are developed for the existence of a polynomial of which the γ are a complete residue system (without real loss of generality $f(\mu) \equiv \gamma_\mu \bmod m$ may be assumed). When $\gamma_1, \cdots, \gamma_m$ are of such character that they may form a residue system, there is always a polynomial of degree smaller than $F(m)$, where $F(m)$ is defined as the smallest integer such that $[F(m)]!$ contains m as a factor.

It is next shown how the coefficients of the polynomial may be restricted so that to a possible residue system corresponds exactly one polynomial.

For a given modulus m the number of possible residue systems is determined.

The methods of proof are based on the examination of polynomials $\varphi(x)$ with rational coefficients which are congruent to zero modulo a given m for all $x = 0, \pm 1, \pm 2, \cdots$ in inf. and on the introduction of "arithmetical sequences modulo m of a given order," which are obtained from ordinary arithmetical sequences of a given order by reducing all elements and also all differences modulo m. Many related questions are treated.

17. Wilczynski's method of discussion of the projective differential geometry of a geometrical configuration is based upon either a certain set of ordinary differential equations or a certain completely integrable system of partial differential equations. A necessary step in the discussion is the construction of a fundamental set of seminvariants. In the cases in which the basic system of differential equations has one dependent variable this construction has been accomplished by the reduction of the system to a canonical form, the independent coefficients of which form are the fundamental set

*Zsigmondy, *Monatshefte für Math. und Physik*, vol. 8 (1897), p. 21.

required. If the basic system of differential equations is a completely integrable system of partial differential equations with one dependent variable, Dr. Nelson shows that this process can be made to yield a pseudo-canonical form, the independent coefficients of which are a simpler fundamental set of seminvariants than the usual process gives.

18. The purpose of the paper by Professor Bliss is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called imbedding theorem for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions. The proofs gain much in simplicity by the use of Peano's matrix notations and his notion of the modulus of a matrix.

19. Professor Ettlinger considers the boundary value problem consisting of the differential equation

$$[K(x, \lambda)u_x]_x - G(x, \lambda)u = 0$$

together with the boundary conditions

$$A_iu(a) - A_i'K(a)u_x(a) - B_iu(b) + B_i'K(b)u_x(b) = 0 \qquad (i = 1, 2),$$

which are not self-adjoint. The types of non-self-adjoint conditions are classified and the existence of at least one characteristic number is established for two important cases.

20. The class of functional equations considered by Professor Carmichael in this note includes linear homogeneous differential, difference and q-difference equations. For the general linear homogeneous differential equation

$$a_0D^ny + a_1D^{n-1}y + \cdots + a_ny = 0, \tag{1}$$

in which D denotes d/dx, a solution y_1 is said to be r-fold if

$y_1, xy_1, \cdots, x^{r-1}y_1$ are all solutions while $x^r y_1$ is not a solution. If r is greater than unity, the solution is said to be repeated. If y_1 is a repeated solution, then it must also satisfy the equation

$$na_0D^{n-1}y + (n-1)a_1D^{n-2}y + \cdots + a_{n-1}y = 0,$$

that is, the equation obtained from (1) by formal differentiation with respect to D. The first elements of the theory of repeated solutions of (1) and a certain more general class of equations thus suggested is developed on a simple postulational basis.

ARNOLD DRESDEN,
Secretary of the Section.

ELEMENTARY INEQUALITIES FOR THE ROOTS OF AN ALGEBRAIC EQUATION.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, October 27, 1917.)

1. LET us write the general algebraic equation in each of the following forms:*

$$x^n = a_1x^{n-1} + a_2{}^2x^{n-2} + a_3{}^3x^{n-3} + \cdots + a_n{}^n,$$

$$(1) \qquad x^n = c_{n1}\alpha_1x^{n-1} + c_{n2}\alpha_2{}^2x^{n-2} + \cdots + c_{nn}\alpha_n{}^n,$$

$$x^n = \beta_1x^{n-1} + \beta_2x^{n-2} + \cdots + \beta_n,$$

where

$$a_i{}^i = c_{ni}\alpha_i{}^i = \beta_i \qquad (i = 1, 2, \cdots, n),$$

and $c_{n1}, c_{n2}, \cdots, c_{nn}$ denote the binomial coefficients for the power n.

If we let X denote the greatest absolute value of a root of equation (1) and let α denote the greatest absolute value of the quantities $|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|$, then, as was shown by Carmichael and Mason,† we have $X \geqq \alpha$, the equality sign

* The fruitful and convenient notation employed in the first equation was suggested to me by my friend and colleague, Dr. A. J. Kempner.

† This BULLETIN, vol. 21 (1914), pp. 14–22. Carmichael and Mason stated the theorem for the equation whose roots are the reciprocals of those of (1).

holding in a special case, namely, for the equation $(x-\alpha)^n=0$. Birkhoff, in the same volume,* reproduced Carmichael and Mason's proof of this theorem and obtained further an upper bound for X, so that on combining the two theorems one has

$$\alpha \leqq X \leqq \frac{\alpha}{2^{1/n}-1}. \tag{2}$$

The theorem in (2) was also proved independently by Jensen.†

Notwithstanding the fact that the upper bound in (2) is attained in a special case, namely, that of the equation $2x^n-(x+\alpha)^n=0$, it is far greater than needful in the case of certain types of equation, as, for instance, the binomial equation $x^n=\alpha^n$. Here the roots are in absolute value equal to α, while the Birkhoff-Jensen upper bound increases indefinitely as n increases. See an extension of the Birkhoff-Jensen result in § 6 below.

The upper bound in (2), as well as several related results, was obtained by Fujiwara‡ by an ingenious method which we shall employ below in the derivation of several theorems.

2. Carmichael and Mason (loc. cit., pages 21–22) also gave an upper bound to the roots of (1), showing indeed that

$$X \leqq \{1+|\beta_1|^2+|\beta_2|^2+\cdots+|\beta_n|^2\}^{\frac{1}{2}}. \tag{3}$$

Probably this bound is never attained. But if we denote by B the second member of (3) and choose ϵ any positive quantity, however small, then we can find equations all of whose roots are in absolute value between $B(1-\epsilon)$ and B. In fact it is sufficient to take a binomial equation $x^n=\beta_n$ and choose β_n sufficiently large. Thus the upper bound in (3) is relatively close, in certain cases at least, when the upper bound in (2) is too great by as large a factor as one pleases. On the other hand, the bound in (3) is unsatisfactory in that it is greater than unity however small the coefficients of the equation may be. Fujiwara§ has employed (3) itself to obtain another bound not having this disadvantage, namely, that given in the relation

$$X \leqq \left\{\frac{n^n}{(n-1)^{n-1}}(|\beta_1|^2+|\beta_2|^2+\cdots+|\beta_n|^2)\right\}^{\frac{1}{2n}}, \tag{4}$$

* Ibid., vol. 21 (1915), pp. 494–495.

† *Nyt Tidsskrift*, vol. 26*A* (1915), pp. 6–13. Some other related results are also given by Jensen.

‡ *Tôhoku Math. Journal*, vol. 10 (1916), pp. 167–171.

§ *Tôhoku Math. Journal*, vol. 8 (1915), pp. 82–85.

this being valid provided that the condition

$$|\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2 < \frac{1}{n-1} \tag{5}$$

is satisfied. He (loc. cit.) has also generalized (3) by showing that, when the roots $x_1, x_2, \cdots, x_n$ of (1) are arranged so that $|x_1| \leqq |x_2| \leqq \cdots \leqq |x_n|$, we have

$$|x_p| \leqq |x_1 x_2 \cdots x_p|^{1/p} \leqq \{1 + |\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2\}^{\frac{1}{2}p}. \tag{6}$$

Kuniyeda* has given interesting generalizations of (3) and (4), showing that

$$X \leqq \{1 + (|\beta_1|^{1+p} + |\beta_2|^{1+p} + \cdots + |\beta_n|^{1+p})^{1/p}\}^{p/(1+p)}, \tag{7}$$

$$X \leqq \left\{ \frac{n^n}{(n-1)^{n-1}} (|\beta_1|^{1+p} + |\beta_2|^{1+p} + \cdots + |\beta_n|^{1+p})^{1/p} \right\}^{p/n(1+p)}, \tag{8}$$

where p denotes any positive number, the latter relation being certainly valid only when

$$(|\beta_1|^{1+p} + \cdots + |\beta_n|^{1+p})^{1/p} \leqq \frac{1}{n-1}. \tag{9}$$

Putting $p = 1$ in (7) and (8), we have the special cases (3) and (4).

The theorem in (3) and certain immediate related results have also been given by S. B. Kelleher† (without reference).

For certain other discussions of inequalities related to those given above reference may be made to papers by Kakeya‡ and Kojima.§ A new proof of some of the results in the latter paper will be given in the sequel.‖

3. The principal object of the present paper is to derive (in §§ 5–10) numerous elementary inequalities for the greatest¶ absolute value X of a root of equation (1). In every case the

* Ibid., vol. 9 (1916), pp. 167–173; vol. 10 (1916), pp. 187–188.

† *Journ. de Math.* (7), vol. 12 (1916), pp. 168–171.

‡ *Tôhoku Math. Journal*, vol. 2 (1912), p. 140; vol. 3 (1913), p. 23.

§ Ibid., vol. 5 (1914), p. 54; vol. 11 (1917), p. 119.

‖ For two other very interesting inequalities on the roots of algebraic equations see Fejér, *Math. Annalen*, vol. 65 (1908), pp. 413–423 and Lévy, *Nouvelles Annales de Math.* (3), vol. 11 (1892), pp. 147–148.

¶ In an obvious manner these yield corresponding inequalities for the *least* absolute value of a root of equation (1).

method is altogether elementary. The results generalize a number of the known inequalities and contain several new ones of interest. It is clear that numerous other inequalities may be obtained by a further use of the elementary methods here employed.

4. The formula given by Carmichael and Mason (loc. cit., pages 21–22) for the exact value of X may be put into a shape more convenient for certain purposes than the determinant form in which they presented it. Let us consider the equation

$$1 - a_1 z - a_2^2 z^2 - \cdots - a_n^n z_n = 0,$$

whose roots are the reciprocals of the roots of (1), and let ξ denote a root of least absolute value of this equation. Write

$$\frac{1}{1 - a_1 z - a_2^2 z^2 - \cdots - a_n^n z^n} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{10}$$

Now, the circle of convergence of the power series in the second member of this equation passes through the point ξ. Its radius of convergence is therefore equal to $|\xi| = 1/X$. Therefore by a well-known property of power series we have

$$X = \limsup_{m=\infty} \sqrt[m]{|c_m|}. \tag{11}$$

But from (10) it is clear that we have

$$1 + (a_1 z + a_2^2 z^2 + \cdots + a_n^n z^n) + (a_1 z + \cdots + a_n^n z^n)^2 + \cdots = 1 + c_1 z + c_2 z^2 + \cdots.$$

Expanding the parenthesis quantities in the first member by means of the multinomial formula and equating coefficients of like powers of z, we have

$$c_m = \Sigma \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1!\, t_2! \cdots t_n!} a_1^{t_1} a_2^{2t_2} a_3^{3t_3} \cdots a_n^{nt_n}, \tag{12}$$

where the summation extends over all the non-negative values $t_1, t_2, \cdots, t_n$ for which $t_1 + 2t_2 + 3t_3 + \cdots + nt_n = m$. Taking (11) and (12) together we have the desired explicit formula for X in terms of the coefficients of equation (1).

Though the value of X given in (11) and (12) is exact it is far too unwieldy to be useful in applications.

By comparing the second member of (12) with the multinomial expansion of $(|a_1| + |a_2| + \cdots + |a_n|)^m$ and making

use of (11) one may readily show that

$$X \leqq |a_1| + |a_2| + \cdots + |a_n|. \tag{13}$$

(A more elementary proof of this formula is given in § 7 below.) It is obvious that the upper bound in (13) is actually attained in the case of binomial equations.

5. Employing the method of Fujiwara,* we start with the inequality

$$\rho^n \leqq |a_1|\rho^{n-1} + |a_2|^2\rho^{n-2} + \cdots + |a_n|^n, \tag{14}$$

which must obviously be satisfied by the absolute value ρ of any root of equation (1). If we suppose that

$$\rho^n > \lambda_k |a_k|^k \rho^{n-k} \qquad (k = 1, 2, \cdots, n),$$

while the positive quantities λ have the sum of their reciprocals not greater than unity, we shall evidently be led to a contradiction with (14). Hence we have the central theorem of Fujiwara's paper, namely,†

$$X \leqq \max \{\lambda_k^{1/k} |a_k|\} \quad (k = 1, 2, \cdots, n), \tag{15}$$

provided that the positive quantities λ satisfy the relation

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \leqq 1. \tag{16}$$

Let us consider the question as to what values of the λ's will make the bound afforded by (15) the lowest possible. It is easy to see that they are the values which render equal the n quantities in the second member of (15) and are such that the equality sign in (16) affords a valid relation. Supposing the λ's chosen in this way, let us write

$$\sigma = \lambda_k^{1/k} |a_k| \qquad (k = 1, 2, \cdots, n).$$

Solving these equations for λ_k and substituting in (16) (with the sign of inequality replaced by the equality sign), we have

$$\sigma^n = |a_1|\sigma^{n-1} + |a_2|^2\sigma^{n-2} + \cdots + |a_n|^n.$$

It is easy to see that this equation has a single positive root. Hence we are led to the following theorem, which we state in the notation of the third form of equation (1):

* *Tôhoku Math. Journal*, vol. 10 (1916), pp. 167–171.

† The theorem obviously remains true if the equality sign is expunged from each inequality in it.

No root of the equation

$$x^n = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$$

is in absolute value greater than the positive root of the equation

$$\sigma^n = |\beta_1|\sigma^{n-1} + |\beta_2|\sigma^{n-2} + \cdots + |\beta_n|.$$

It is evident that this bound is actually attained in the case of any equation in which the β's are positive or zero. Moreover, it is clear that this is the lowest bound that can be obtained by the direct use of equation (14). Furthermore, it is obvious that a close approximation to this bound is readily obtained in the case of any numerical equation.

6. From (16) it follows that $\lambda_k \geqq 1$; and thence, when (16) is satisfied, we see that

$$X \leqq \max \{\lambda_k^{l_k} |a_k|\} \quad (k = 1, 2, \cdots, n), \tag{17}$$

where l_k is any real quantity not less than $1/k$.

Applying this to an equation of $t+1$ terms, we have the following result:

The maximum absolute value X of a root of the equation

$$x^n = \gamma_1^{n-n_1} x^{n_1} + \gamma_2^{n-n_2} x^{n_2} + \cdots + \gamma_{t-1}^{n-n_{t-1}} x^{n_{t-1}} + \gamma_t^n, \tag{18}$$

where $n, n_1, \cdots, n_{t-1}$ form a decreasing sequence of positive integers, satisfies the relation

$$X \leqq \max \{\lambda_k^{1/k} |\gamma_k|\} \quad (k = 1, 2, \cdots, t), \tag{19}$$

provided that the positive quantities λ are subject to the condition

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_t} \leqq 1. \tag{20}$$

If we consider the question as to what values of the λ's will make the bound afforded by (19) the lowest possible we shall be led by the method of § 5 to the following theorem (giving such lowest bound):

No root of equation (18) *is in absolute value greater than the positive root of the equation*

$$s^t = |\gamma_1| s^{t-1} + |\gamma_2|^2 s^{t-2} + |\gamma_3|^3 s^{t-3} + \cdots + |\gamma_t|^t.$$

Taking $\lambda_k = t$ in (19), we see that we have for equation

(18) the relation

$$X \leqq \max \{t^{1/k}|\gamma_k|\} \quad (k = 1, 2, \cdots, t),$$

a result given by Cauchy for the case $t = n$.

Next, let us set

$$\lambda_k = \frac{1}{c_{tk}(\sqrt[t]{2} - 1)^k}.$$

Then (20) is a valid relation. Hence we have through (19) the inequality

$$X \leqq \max \left\{ \frac{|\gamma_k|}{c_{tk}(\sqrt[t]{2} - 1)} \right\} \quad (k = 1, 2, \cdots, t).$$

The special case $t = n$ is the Birkhoff-Jensen upper bound given in (2). The method of proof is that employed for this special case by Fujiwara in the *Tôhoku Mathematical Journal*, volume 10. For an equation of high degree and relatively few terms the bound is much closer than that of Birkhoff and Jensen.

7. In relation (17) we may choose special values of the quantities l_k so as to obtain the relation

$$X \leqq \operatorname{Max} \{\lambda_1|a_1|, \lambda_2|a_2|, \cdots, \lambda_{k_1}|a_{k_1}|, \lambda_{k_1+1}{}^{\frac{1}{2}}|a_{k_1+1}|, \cdots, \lambda_{k_2}{}^{\frac{1}{2}}|a_{k_2}|, \lambda_{k_2+1}{}^{\frac{1}{3}}|a_{k_2+1}|, \cdots, \lambda_n{}^{1/t}|a_n|\},$$

provided that the positive quantities λ satisfy (16). If we consider the question as to what values of the λ's will make the bound afforded by this relation the lowest possible we shall be led by the method of § 5 to the following theorem (giving such lowest bound):

No root of equation (1) *is in absolute value greater than the positive root of the equation*

$$r^t = (|a_1| + \cdots + |a_{k_1}|)r^{t-1} + (|a_{k_1+1}|^2 + \cdots + |a_{k_2}|^2)r^{t-2} + \cdots + (|a_{k_{t-1}+1}|^t + \cdots + |a_n|^t). \tag{21}$$

It is clear that one may prove in a similar way that

No root of equation (1) *is in absolute value greater than the positive root of the equation*

$$r^n = (|a_1| + \cdots + |a_{k_1}|)r^{n-1} + (|a_{k_1+1}|^{k_1+1} + \cdots + |a_{k_2}|^{k_1+1})r^{n-k_1-1} + \cdots + (|a_{k_{t-1}+1}|^{k_{t-1}+1} + \cdots + |a_n|^{k_{t-1}+1})r^{n-k_{t-1}-1}. \tag{22}$$

In the former of the last two results, take $t = 1$; then we have

$$X \leqq |a_1| + |a_2| + \cdots + |a_n|,$$

a relation obtained in § 4 above by less elementary means.

In (21) let $t = 2$ and write

$$|a_1| + \cdots + |a_{k_1}| = s_1, \quad |a_{k_1+1}|^2 + \cdots + |a_n|^2 = s_2.$$

Then we see that

$$X \leqq \tfrac{1}{2}\{s_1 + \sqrt{s_1^2 + 4s_2}\}.$$

In particular, if $a_1 = 0$ we may take $k_1 = 1$ and show that

$$X \leqq \sqrt{|a_2|^2 + |a_3|^2 + \cdots + |a_n|^2}.$$

More generally, if $a_1 = 0 = a_2 = \cdots = a_{\mu-1}$, we have readily from (22) the relation

$$X \leqq \sqrt[\mu]{|a_\mu|^\mu + |a_{\mu+1}|^\mu + \cdots + |a_n|^\mu}.$$

We may also apply to (21) and (22) the theorems already proved for equation (1) and thus obtain other results for equation (1) itself. Thus through use of (15) and (22) we see that

$$X \leqq \max\{\lambda_1(|a_1| + \cdots + |a_{k_1}|), [\lambda_2(|a_{k_1+1}|^{k_1+1} + \cdots + |a_{k_2}|^{k_1+1})]^{1/(k_1+1)}, \cdots\}$$

provided that the positive quantities λ satisfy relation (20).

8. In (19) let us take

$$\lambda_k = \frac{1}{|\gamma_k|^k}(|\gamma_1| + |\gamma_2|^2 + \cdots + |\gamma_t|^t).$$

Then (20) is satisfied and from (19) we see that for equation (18) we have

$$X \leqq \max\{|\gamma_1| + |\gamma_2|^2 + \cdots + |\gamma_t|^t\}^{1/k} \quad (k = 1, 2, \cdots, t).$$

Hence for equation (18) we have

$$X \leqq \left\{\sum_{i=1}^{t} |\gamma_i|^i\right\}^{1/t} \text{ if } \sum_{i=1}^{t} |\gamma_i|^i \leqq 1;$$

(23)

$$X \leqq \sum_{i=1}^{t} |\gamma_i|^i \qquad \text{if } \sum_{i=1}^{t} |\gamma_i|^i \geqq 1.$$

The method of proof here employed is identical with that

used by Fujiwara (loc. cit.) for the special case $t = n$. From the first relation in (23) we see readily that

$$X \leqq \left\{ t \sum_{i=1}^{t} |\gamma_i|^{2i} \right\}^{1/2t} \quad \text{if} \quad \sum_{i=1}^{t} |\gamma_i|^i \leqq 1,$$

since

$$\left(\sum_{i=1}^{t} |\gamma_i|^i \right)^2 \leqq t \sum_{i=1}^{t} |\gamma_i|^{2i}$$

as one may readily prove.

9. The theorem of Fujiwara stated in relation (15) will also yield the upper bound given in (3). For the third form of equation (1) relation (15) may be written

$$X \leqq \max \{\lambda_k |\beta_k|\}^{1/k} \quad (k = 1, 2, \cdots, n),$$

the positive quantities λ being again subject to relation (16). For the proof of (3) it is obviously sufficient to show that (16) is satisfied if the λ's have the values

$$\lambda_k = \frac{1}{|\beta_k| \nu^k}, \quad \text{where} \quad \frac{1}{\nu} = \{1 + |\beta_1|^2 + \cdots + |\beta_n|^2\}^{1/2}.$$

For the moment denote the first member of (16) by S. Then we have

$$\begin{aligned} S &= |\beta_1|\nu + |\beta_2|\nu^2 + \cdots + |\beta_n|\nu^n \\ &\leqq \{|\beta_1|^2 + \cdots + |\beta_n|^2\}^{1/2} \{\nu^2 + \nu^4 + \cdots + \nu^{2n}\}^{1/2} \\ &\leqq \left\{ \frac{1}{\nu^2} - 1 \right\}^{1/2} \{\nu^2 + \nu^4 + \cdots + \nu^{2n}\}^{1/2} \\ &\leqq \{1 - \nu^{2n}\}^{1/2} < 1, \text{ since } \nu < 1. \end{aligned}$$

Hence (16) is satisfied and (3) is proved.

Since (3) is proved by aid of (15) it is clear that it can never afford a lower bound for X than that given in the italicized theorem of § 5.

In a similar way one may prove the generalization (7) due to Kuniyeda. The method is essentially that of Kuniyeda's memoir. It is easy to see further that (7) can never afford a lower bound for X than that given in the italicized theorem in § 5.

10. From the third form of equation (1) we have

$$|x|^{n-k}|x^k - \beta_1 x^{k-1} - \cdots - \beta_k| \leqq |\beta_{k+1}| \cdot |x|^{n-k+1} + \cdots + |\beta_n|.$$

Hence if

$$|x^k - \beta_1 x^{k-1} - \cdots - \beta_k| \geqq r \quad (r > 0),$$

we have

$$|x|^{n-k} \leqq \frac{1}{r}|\beta_{k+1}| \cdot |x|^{n-k-1} + \cdots + \frac{1}{r}|\beta_n|.$$

Therefore the last inequality will certainly be satisfied if

$$|x|^k \geqq |\beta_1| \cdot |x|^{k-1} + \cdots + |\beta_{k-1}| \cdot |x| + (|\beta_k| + r).$$

Thence it is easy to see that

The greatest absolute value X of a root of equation (1) *satisfies the relation*

$$X \leqq \max\{\rho_1, \rho_2\},$$

where ρ_1 and ρ_2 are the positive roots of the equations

$$\rho^k = |\beta_1|\rho^{k-1} + \cdots + |\beta_{k-1}|\rho + (|\beta_k| + r), \tag{24}$$

$$\rho^{n-k} = \frac{1}{r}|\beta_{k+1}|\rho^{n-k-1} + \cdots + \frac{1}{r}|\beta_n|, \tag{25}$$

respectively, r being any positive quantity and k any integer less than n.

This theorem itself may obviously be applied to either or both of equations (24) and (25); so that an upper bound to X may be obtained in terms of the solutions of equations of as low degree as one wishes. Thus, as a special case, we may take the equations to be linear and thus obtain the following result (found by Kojima* in a less elementary way):

$$X \leqq \max\left\{|\beta_1| + r_1, \frac{|\beta_k|}{r_1 r_2 \cdots r_{k-1}} + r_k \quad (k = 2, \cdots, n-1), \frac{|\beta_n|}{r_1 r_2 \cdots r_{n-1}}\right\},$$

where $r_1, r_2, \cdots, r_{n-1}$ are any positive quantities.

* *Tôhoku Math. Journal*, vol. 11 (1917), pp. 119–127. The first two special cases noted had been given earlier by the same author, ibid., vol. 5 (1914), pp. 54–60.

Taking each r equal to unity, we have the interesting special case

$$X \leqq \max\{1 + |\beta_1|, 1 + |\beta_2|, \cdots, 1 + |\beta_{n-1}|, |\beta_n|\}.$$

Again, it is easy to show also that

$$X \leqq \max\{|\alpha| + |\beta|, |\beta| + |\gamma|, \cdots, |\lambda| + |\mu|, |\mu|\},$$

where the quantities $\alpha, \beta, \cdots, \mu$ are defined by the relations

$$\alpha = \beta_1, \beta^2 = \beta_2, \beta\gamma^2 = \beta_3, \beta\gamma\delta^2 = \beta_4, \cdots, \beta\gamma\cdots\lambda\mu^2 = \beta_n.$$

Through other special choices of the quantities r numerous rather elegant special inequalities may be obtained, several of which are given explicitly by Kojima (loc. cit.).

In case some of the coefficients of equation (1) are zero it may be preferable to employ the second italicized theorem in § 6 and apply to it the principal theorem of this section (or the special cases of the latter).

It is clear that other general formulas may be found for upper bounds to X by employing a sequence of equations each of which is linear or quadratic, such sequence arising by the repeated application of the theorem of this section.

UNIVERSITY OF ILLINOIS,
October, 1917.

THE SOLUTION OF THE WAVE EQUATION BY MEANS OF DEFINITE INTEGRALS.

BY PROFESSOR H. BATEMAN.

THE wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \tag{1}$$

is the oldest member of the family of partial differential equations, and although he was born without the second and third terms he soon acquired these and played a prominent part in mathematical physics at a time when very few partial differential equations had become famous. With the advent of the mathematical theory of elasticity and Maxwell's electromagnetic theory of light he gained a new lease of life and more

than doubled his vitality, so that his power is strongly felt even at the present time. In a spirit of reverence for age and valuable service we can still learn much by consulting the old pioneer; not only is he able to throw light on the much discussed subject of the nature of electricity and the general theory of partial differential equations, but he is leaving us a legacy of unsolved problems concerning the diffraction of sound and electric waves and these will tax the ingenuity of mathematicians for years to come. It is often said that Maxwell's equations do not tell us anything about the structure of the æther, but this view may not be correct. Although the equations themselves do not give any clear indications of the properties of the æther, yet the solutions of the equations may give some information. It must be realized that when a physical problem is reduced to differential equations some of the features of the problem are eliminated in the process and very often these features are of great physical interest. Unfortunately we cannot retrace the process of elimination and the only process of reconstruction open to us is to make an exhaustive study of the solutions of the differential equations and to pay particular attention to cases in which a very general solution is built up from elementary solutions of a particularly simple type.

In the solution of diffraction problems the modern tendency seems to be to make use of integrals taken along complex paths in representing the required solutions of the wave equation.* This method is very powerful and beautiful, but to obtain a visualization of the physical processes which actually take place it seems desirable to make use of integrals taken along real paths and some progress has already been made in this direction. Again in A. W. Conway's derivation† of the potentials for a moving point charge of electricity use is made of contour integrals of type

$$(2)\quad \int \frac{f(\tau)d\tau}{[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2-c^2(t-\tau)^2},$$

* See, for instance, A. Sommerfeld, *Math. Ann.*, Bd. 45 (1894), Bd. 47 (1896); *Zeitschr. für Math. u. Phys.*, Bd. 46 (1901); H. M. Macdonald, Electric Waves (1902); *Proc. London Math. Soc.*, ser. 2, vol. 14 (1915), p. 410; H. S. Carslaw, *Proc. London Math. Soc.*, ser. 1, vol. 30 (1898), p. 121; *Math. Ann.*, Bd. 75 (1914), p. 133; T. J. I'A. Bromwich, *Proc. London Math. Soc.*, ser. 2, vol. 14 (1915), p. 450; F. J. W. Whipple, ibid., vol. 16 (1917), p. 94.

† *Proc. London Math. Soc.*, ser. 2, vol. 1 (1903).

but it has been shown recently* that a similar derivation is possible by taking the integral (2) between certain consecutive real limits τ_1 and τ_2, each of which is defined by a pair of equations of type

$$[x - \alpha(\theta)]^2 + [y - \beta(\theta)]^2 + [z - \gamma(\theta)]^2 = c^2(t - \theta)^2, \quad t \geqq \theta,$$

$$[\alpha(\theta) - \xi(\tau)]^2 + [\beta(\theta) - \eta(\tau)]^2 + [\gamma(\theta) - \zeta(\tau)]^2 = c^2(\theta - \tau)^2, \quad \theta \geqq \tau.$$

It is easy to verify that the definite integral is in this case a wave function;† it is also easy to verify that an integral of the above type represents a wave function when the limits are functions of type τ which is defined by an equation such as

$$l(\tau)[x - \xi(\tau)] + m(\tau)[y - \eta(\tau)] + n(\tau)[z - \zeta(\tau)] = c(t - \tau),$$

where

$$[l(\tau)]^2 + [m(\tau)]^2 + [n(\tau)]^2 \equiv 1.$$

This brings us to the consideration of the following problem: If $f(x, y, z, t, \theta)$ is a wave function for all values of the parameter θ, the definite integral

$$V = \int_{\theta_1}^{\theta_2} f(x, y, z, t, \theta)d\theta$$

is certainly a wave function when θ_1 and θ_2 are suitable constants, but it may also be a wave function when θ_1 and θ_2 are certain functions of x, y, z and t. The problem is to find a suitable pair of functions θ_1 and θ_2 when $f(x, y, z, t, \theta)$ is given.

Let us commence by considering the case when f is a wave function of type‡ $\gamma F(\alpha, \beta)$ where F is an arbitrary function of its two arguments and α, β, γ are functions of x, y, t and z. This type of wave function may be regarded as more fundamental even than the familiar function

$$[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - c^2(t - \tau)^2]^{-1},$$

* H. Bateman, *Tohôku Math. Journal*, vol. 13.

† The analogous solution of the wave equation in two dimensions may be used to solve the problem of the diffraction by a half plane of the waves from a moving line source of sound, the line being parallel to the edge of the diffracting plane.

‡ The problem of finding all possible forms for α, β and γ is solved in this BULLETIN, vol. 21 (1915), p. 299, and in *Amer. Journal of Mathematics*, April, 1915. We may write

$$z - ct = \phi(\alpha, \beta) + (x + iy)\chi(\alpha, \beta),$$
$$z + ct = \psi(\alpha, \beta) - (x - iy)/\chi(\alpha, \beta).$$

which gives rise to the integrand in (2). In electromagnetic theory, for instance, if $\boldsymbol{E}$ and $\boldsymbol{H}$ denote electric and magnetic intensities respectively, the æthereal field in which the complex vector $\boldsymbol{M} \equiv \boldsymbol{H} + i\boldsymbol{E}$ is represented by the expressions

$$M = F(\alpha, \beta)(\nabla\alpha \times \nabla\beta) \equiv \frac{i}{c} F(\alpha, \beta) \left[\frac{\partial\beta}{\partial t} \nabla\alpha - \frac{\partial\alpha}{\partial t} \nabla\beta \right]$$

is a *pure radiant field*, while the quantities α and β are constant along the rays which are the rectilinear paths of the energy. It has been shown that the field of a point charge of electricity can be regarded as the limiting case of two superposed radiant fields of the above description in each of which electricity flows with the velocity of light along certain pairs of singular rays. The singular rays in the two fields overlap and there is a cancelling of electricity except on two segments which shrink to a point as the limit is approached. This suggests that the æther may consist of particles of electricity and electric doublets travelling along straight lines with the velocity of light. In this connection it may be worth while to point out that the rays are bicharacteristics* of the partial differential equations while α and β are functions which satisfy the partial differential equation of the characteristics of equation (1).

Let us now make α and β depend on a parameter θ as well as on x, y, t and z; forming the definite integral

$$M = \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta)(\nabla\alpha \times \nabla\beta) d\theta$$

$$\equiv \frac{i}{c} \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta) \left(\frac{\partial\beta}{\partial t} \nabla\alpha - \frac{\partial\alpha}{\partial t} \nabla\beta \right) d\theta$$

and writing

$$\frac{\partial\alpha}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial\alpha}{\partial z}\frac{\partial\theta}{\partial z} - \frac{1}{c^2}\frac{\partial\alpha}{\partial t}\frac{\partial\theta}{\partial t} \equiv (\alpha, \theta),$$

we find that

$$\text{rot } M + \frac{i}{c}\frac{\partial M}{\partial t} \equiv \{(\beta, \theta)\nabla\alpha - (\alpha, \theta)\nabla\beta\} F(\alpha, \beta, \theta) \Big|_{\theta=\theta_1}^{\theta=\theta_2}.$$

The right-hand side is clearly zero if θ_1 and θ_2 are solutions of

* The theory of bicharacteristics is expounded in Hadamard's Propagation des Ondes.

the two equations

$$(\alpha, \theta) = 0 \quad \text{and} \quad (\beta, \theta) = 0.$$

Now $(\alpha, \alpha) = 0$, $(\alpha, \beta) = 0$ and $(\beta, \beta) = 0$, hence the above equations may be satisfied by taking θ_1 and θ_2 to be roots of the equations

$$G_1(\alpha, \beta, \theta) = 0 \quad \text{and} \quad G_2(\alpha, \beta, \theta) = 0$$

respectively.* It should be noticed in passing that θ_1 and θ_2 are solutions of the characteristic equation $(\theta, \theta) = 0$. Since

$$\text{rot } M + \frac{i}{c}\frac{\partial M}{\partial t} = 0$$

it follows that each component of M satisfies the wave equation and so we have an important class of wave functions represented by definite integrals in which the limits are functions of x, y, z and t. It should be noticed that if we substitute a definite integral of type

$$V = \int_{\theta_1}^{\theta_2} \gamma F(\alpha, \beta, \theta) d\theta$$

in the wave equation we obtain a system of partial differential equations for θ_1 and θ_2 and these are difficult to solve directly. The above method, however, indicates the nature of the complete solution of these equations. It also emphasizes the advantage which is gained by using a system of partial differential equations of the first order instead of the second order equation (1). This plan might be adopted with advantage in other cases; it may be recalled also that the first order equations may be used more profitably than (1) in the determination of all the functions of type α and β.

The theorem that has just been proved is of wide application. We know for instance that a wave function of type $\gamma F(\alpha, \beta)$ may be obtained by writing

$$\gamma = \frac{1}{R}, \quad \alpha = t - \frac{R}{c}, \quad \beta = \frac{x - iy}{R + z},$$

where R is the distance from the origin. We may generalize this solution by shifting the origin to the point $\xi(\theta)$, $\eta(\theta)$,

* G_1 and G_2 are arbitrary functions of α, β and θ.

$\zeta(\theta)$ and we then have the result that if

$$r^2 = [x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2,$$

$$\alpha = t - \frac{r}{c}, \quad \beta = \frac{x - \xi(\theta) - i[y - \eta(\theta)]}{r + [z - \zeta(\theta)]},$$

the definite integral

$$V = \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta) \frac{d\theta}{r}$$

satisfies the wave equation, provided θ_1 and θ_2 are roots of the equations

$$G_1(\alpha, \beta, \theta) = 0 \quad \text{and} \quad G_2(\alpha, \beta, \theta) = 0$$

respectively. The case in which F, G_1 and G_2 are independent of β is of special interest. V may then be regarded as the velocity potential of a chain of sources of sound each of which is only active for a certain interval of time which may be different for different sources.

THROOP COLLEGE OF TECHNOLOGY,
PASADENA,
October 6, 1917.

LITERATURE OF PURE MATHEMATICS.

Historical Introduction to Mathematical Literature. By G. A. MILLER. New York, Macmillan, 1916. 14 + 302 pp. Price $1.60.

IT is difficult to overestimate the extent of inspiration which may emanate from interesting exposition of problems and wonders of science in a form intelligible to those who are not deeply versed in the subject. How much richer must be the intellectual outlook of thousands throughout the world who have read: Science and Hypothesis, The Value of Science, and Science and Method! Is it hard to believe that the future historian may some day tell us that the very notable stage of advancement of astronomy in America in 1910 was not a little due to Simon Newcomb's remarkable gifts of popularization of his subject, exercised through written and spoken word in magazine, book, society, and congress during

a long series of years—educating and inspiring the people to a frame of mind which called for organization, equipment, and cooperation greater than had been?

For many years Professor Miller has been one of a small group of Americans who have very frequently written and spoken with authority in the interests of the mathematical amateur and inquiring general student. There can be little doubt of the success of their efforts to arouse in America more widespread enthusiasm and deeper appreciation of the values involved in the study of mathematics, through discussion of some of its historical, philosophical, geometrical, and analytical aspects. In this connection apart from expository articles (more particularly in the study of groups) Professor Miller has published many articles and addresses attractive not so much on account of their stimulating style, for in this they are sometimes lacking, but rather on account of their timely and highly informational content, very clearly, suggestively, and thoughtfully expressed. We welcome then this unique work,* with similar characteristics, in which much that had been published is coordinated, developed, and expanded, and combined with new features.

We are told that the volume is based on a course of lectures which were designed to supplement the regular work of mathematical courses, and that in it the author (1) "aimed to meet the needs of a textbook for synoptic and inspirational courses which can be followed successfully by those who have not had extensive mathematical training"; and (2) provided a work which "may also be used as a textbook for a first course in the history of mathematics, especially by those teachers who believe with its author that such a first course should largely concern itself with recent mathematical events and developments." The author states that in his presentation he attempted to guide the mathematical student to "points from which he can overlook domains of considerable extent in order that he may be able to form a somewhat independent judgment as regards the regions which he might like to examine more closely." Let us now consider the work in some detail.

The "General Observations" (35 pages) of the first chapter

* G. Loria's Guida allo Studio della Storia delle Matematiche (Milano, 1916), which was published almost simultaneously with this work has very different ideals. See the review by D. E. Smith in this BULLETIN, 1916, vol. 23, pp. 136–139.

are made under the headings: Changes during the nineteenth century; history of mathematics; developed parts of the history of mathematics; periods in the history of mathematics; usefulness of the history of mathematics; first decade and a half of the twentieth century; American mathematics. Since details of purely domestic interest are recorded in connection with the references to the American Mathematical Society, it is a wonder that one fact of international interest is not mentioned, namely, that since 1913 the society has enjoyed "the possibly unique distinction of holding almost simultaneous meetings in different cities."*

The second chapter on "Mathematical Literature in General" is somewhat longer. In the first section on "Types of recent mathematical literature" the discussion is mainly confined to English, French and German periodicals. Except for slips in the dates of the *Cambridge Mathematical Journal* and of the *Cambridge and Dublin Mathematical Journal*, the statements concerning English periodicals are accurate; but surely wrong inferences may be drawn from the two pages devoted to *The Ladies' Diary* (1704–1840) and the *Gentleman's Diary or the Mathematical Repository* (1741–1840).† We are told that the latter "may be regarded as a scientific mathematical journal and it contains a number of important problems and solutions relating to elementary mathematics." And then: "These journals [the diaries] did much to keep alive in England a general interest in mathematics, and they contain a number of theorems relating to elementary mathematics which were rediscovered in later times." Since, for the period they cover, no other English mathematical journal is mentioned, one natural inference from the above would be that there was no other of equal importance. And yet, for example in the eighteenth century‡ there were two journals devoted *wholly*§ to mathematics and certainly of greater scientific significance. The one, *Miscellanea Curiosa Mathematica*,‖ contained William Chapple's notable paper (1746) with its results concerning the distance between the centers of the

* In this BULLETIN, January, 1914, vol. 20, p. 170.

† No reference is made to the amalgamation of the almanacs into *The Lady's and Gentleman's Diary* (1841–1871).

‡ Similar illustrations could be given concerning the nineteenty century.

§ In the latter part of the century not more than a third of the *Diaries* was of a mathematical nature.

‖ Fourteen numbers published in London, 1745–1754.

inscribed and circumscribed circles of a triangle, and other observations anticipating formulas of Euler* and discussions by Poncelet, Steiner, and Jacobi. The other journal, *The Mathematical Repository*,† contains the first enunciation (March, 1799) of the well-known theorem concerning Wallace's line.‡

On page 44 reference might be made to De Morgan's sketch of the "Mathematical Society of Spitalfields" in his Budget of Paradoxes.

Prince Boncompagni's *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche* is listed in a section on "special historical periodicals," and it is stated that a "peculiarity of this periodical is that very extended references are given even when they relate to matters where the literature s easily accessible, and it is probable that these extensive references have tended to decrease the use of the journal, since they are often so numerous as to burden the reader." A much less trivial "peculiarity," and a source of real exasperation to the historical inquirer, is due to the fact that there are probably few (perhaps not even two) sets of the *Bullettino* which are exactly alike throughout. As the periodical was set up and printed in his palace Boncompagni frequently stopped the press after some numbers were printed in order to introduce additions and changes. In this way it is said that several different editions of the same number may be found.§

In the section on "Encyclopedias and other works of reference" a misleading statement has been made in connection with "the Encyclopédie Méthodique Mathématique, in three volumes, by d'Alembert and others, 1784–89." My edition is in at least four volumes, the last of which, Recueil de Planches du Dictionnaire Encyclopedique des Mathématiques, was published in M.DCC.XCVII. I have written "at least" because there are three more volumes published with the following titles: (1) Dictionnaire des Jeux, faisant suite au

* *Novi Commentarii . . . Academiae . . . Petropolitanae*, tome 11 (1765) 1767, pp. 103–123.

† London, 1795–1835.

‡ Proposition I, in a paper by William Wallace entitled "Mathematical Lucubrations," vol. II, p. 111. It is implied in a parabola theorem which he published in the fourth number of the *Repository* (October, 1797).

§ Cf. *Isis*, June, 1914, tome II, pp. 133–134. This reference is to an article by G. Sarton containing an annotated list of "Soixante-deux revues et collections relatives à l'histoire des sciences," to which attention might naturally be drawn in such a section as that which we are considering.

Tome III des Mathématiques, 1792; (2) Dictionnaire des Jeux familiers ou des Amusemens de Société, faisant suite au Dictionnaire des Jeux, annexé au tome III des Mathématiques, An V [1797]; (3) Dictionnaire des Jeux mathématiques contenant l'Analyse, les Recherches, les Calculs, les Probabilités et les Tables numériques, publiés par plusieurs célèbres Mathématiciens, relativement aux Jeux de Hasard et de Combinaisons, et suite du Dictionnaire des Jeux, an VII [1799].

In this same section we find also: "The most useful work of reference for the purpose of finding the periodical literature of the nineteenth century is the Subject Index of the Royal Society of London Catalogue of Scientific Papers, volume 1, 1908. . . . Since the beginning of the twentieth century this work is supplemented annually by volume A of the International Catalogue of Scientific Literature." And why not give reference to volume 2, 1909 (Mechanics), to volume 3, 1912–14 (Physics, but with much of value for the mathematician), to volumes B (Mechanics), E (Astronomy), J (Geography, which includes geodesy) and C (Physics)? The fact of the matter is that our author disregards applied mathematics practically altogether, so that a more accurate title of his book would be "Historical Introduction to the Literature of Pure Mathematics."

In another section of the second chapter, on "Mathematical tables," a reference to the work edited by Horsburgh* would have usefully supplemented that to the 1873 Report of the British Association for the Advancement of Science. The following statement is perhaps a trifle misleading: "In 1911 H. Andoyer of the University of Paris published a table giving the logarithms of the trigonometric functions . . . under the title Nouvelles Tables Trigonométriques Fondamentales, as it is expected to become fundamental for other tables." The volume in question was the first of three volumes and it consists mainly of a set of logarithmic tables for sines, cosines, tangents and cotangents. The second and third† volumes (1915–16) are principally occupied with tables of natural trigonometric functions. It seems strange to find no

* Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition (1914).

† This volume may have appeared after Professor Miller's book was out.

mention in this section of the outstanding works by Crelle,* Peters,† and Bauschinger.‡

Other topics of the chapter are: Mathematical societies, international mathematical congresses, periodicals on periodicals and books on books, and collected works.

The third chapter, entitled "General Historical Questions Relevant to Mathematics," includes discussion of such subjects as: Definitions of the term mathematics, divisions and subdivisions of mathematics, some dominant concepts, notation and terminology, errors in mathematical literature, and living mathematicians. As the 1908 and 1916 editions of the Index du Répertoire Bibliographique des Sciences Mathématiques have been published by Delsman in Amsterdam, a change is desirable in a statement on page 89. Another instance, among several, of national rather than international outlook occurs on page 123 where, by indirection, H. H. Stephenson's Who's Who in Science (International) which passed through several editions (1912–1914) is condemned as a source of information as to "mathematical standing," while American Men of Science is characterized as a "much more useful type of work."

In the next three chapters fundamental developments in arithmetic, geometry and algebra are discussed. Arithmetical topics are: Existence of an infinite number of prime numbers with Euclid's proof, sieve of Eratosthenes, irrational quantities and numbers, fundamental operations of arithmetic, systems of notation, and Fermat's last theorem. Is it not misleading to give, without comment, 1228 as the date of the "noted Liber Abaci by Leonardo of Pisa"?

The headings for the chapter in geometry are: The Pythagorean theorem, area of the circle, area and volume of the sphere, regular geometric forms, the triangle. It is recalled that the French and the Germans frequently refer to the

* A. Crelle, Calculating tables giving the products of every two numbers from one to one thousand and their application to the multiplication and division of all numbers above one thousand. New edition by O. Seeliger, 1908.

† J. Peters: 1. New calculating tables for multiplication and division, by all numbers of from one to four places. 1909. 2. Siebenstellige Logarithmentafel der trigonometrischen Funktionen, für jede Bogensekunde des Quadranten. 1911.

‡ J. Bauschinger and J. Peters, Logarithmic-trigonometrical tables with eight decimal places containing the logarithms of all numbers from 1 to 200,000 and the logarithms of the trigonometric functions for every sexagesimal second of the quadrant. 2 vols. 1910–11.

Pythagorean proposition as "the asses' bridge." Other names might have been mentioned: "Theorem of the married woman" by the Greeks; "Chaise of the little married woman" by the Hindus (Bhascara); and "figure of the bride" by the Persians.* The first edition of J. J. I. Hoffman's collection of proofs of the proposition is referred to; a "zweyte verbesserte und mit einigen neuen Beweisen vermehrte Ausgabe" was published in 1821.

On pages 175–6 it is stated that Gauss proved (Disquisitiones Arithmeticæ, section 7) that "a necessary and sufficient condition that a regular polygon of a prime number of sides can be inscribed in a circle by means of ruler and compass is that this prime be of the form $2^m + 1$." If the reference to the Disquisitiones had been omitted no criticism could have been made of this statement; but as it is, incorrect inferences are likely to be drawn.†

A reference is given to Casey's Sequel to Eculid, 1888, for information concerning the geometry of the triangle. It may be noted that it is this particular edition alone which contains the supplementary chapters on this subject.

Some of the topics in the chapter on Developments in Algebra are: Fundamental theorem of algebra, determinants, numerical equations, domains of rationality, invariants and binomial theorem.

The seventh and last chapter contains brief sketches of twenty-five prominent deceased mathematicians: Euclid, Archimedes, Apollonius of Perga, Diophantus of Alexandria, Viète, Descartes, Fermat, Newton, Leibniz, Euler, Lagrange, Gauss, Cauchy, Steiner, Abel, Hamilton, Galois, Sylvester, Weierstrass, Cayley, Kronecker, Hermite, Cremona, Lie, and Poincaré. In the sketch of Euler it is stated that "he arrived, in 1748, at the celebrated formula

$$e^{ix} = \cos x + i \sin x."$$

i for $\sqrt{-1}$ appears to have been first used by Euler in 1777. In 1722 Roger Cotes gave the equivalent, in words, of $\sqrt{-1}\varphi = \log(\cos\varphi + i \sin\varphi)$. In the sketch of Gauss reference is made to his proof of the fundamental theorem of

* Cf. E. Lucas, Récréations mathématiques, tome II, 1883, or 1896, p. 130; E. Fourrey, Curiosités géométriques, Paris, 1907, p. 64.

† In more than one place in Klein's Famous Problems of Geometry misstatements occur in this connection.

algebra, to various parts of the Disquisitiones Arithmeticæ, to his work on the theory of orbits and theory of errors, and to his geometric theory of complex numbers. His fundamental monograph in the theory of surfaces is not mentioned, even though the best edition of it was published by Princeton University. It is not accurate to refer to even the major portion of Euclid's book On Divisions (of figures) as "still extant" (page 219). And finally, in the sketches of Hermite and Poincaré our author does not seem to be sufficiently clear as to the use of some French terms: (1) Hermite "became professor also at the École Polytechnique and at the Faculté des Sciences" (page 261); for "Faculté des Sciences" substitute "Université de Paris"; (2) Poincaré "was elected a member of l'Académie des Sciences" (page 269), and "was received as member of l'Académie Française"* (page 271), and "received two prizes from the Paris Academy" (page 272); after each of the first two expressions quoted add "de l'Institut de France," although there is no ambiguity in the second case; for "Paris Academy" (which may be interpreted as having an entirely different meaning from the one intended†) substitute "Institut de France" or "Académie des Sciences de l'Institut de France." It is surely of interest to bring out more clearly the fact that Poincaré was a member of these two Académies of the five composing the Institut de France—the greatest honor which France has in her gift for the élite of her scientists. I am under the impression that J. L. F. Bertrand was the only other scientist who has been similarly honored.

An appendix of 15 pages contains an annotated list of bibliographies and encyclopedias, histories, and books on teaching and philosophy of mathematics. The reviewer sees no advantage in giving partial indications of the first, instead of the last, editions of the works mentioned; and even in this respect there is a slip in connection with Cantor's Vorlesungen. The current edition of the Encyklopädie der Elementar-Mathematik by Weber and Wellstein contains four, not three, volumes (page 280).‡

* This Académie has a "président" but not a "directeur" (p. 271).

† *Cf.* this BULLETIN, March, 1910, vol. 16, p. 329.

‡ There are fairly obvious typographical errors on page 87, line 13; page 130, line 2 from bottom; page 289, line 17; and page 293, line 5. The spelling of the name Čebyšëv on page 133, line 5, does not agree with that later on the page or in the index; similarly Lobačevskÿ on pages

The Author Index and Subject Index occupy about ten pages.

To sum up: Professor Miller has written a valuable, original, and very entertaining book, containing much out-of-the-way information difficult of access elsewhere. It deals only with pure mathematics, and in this field emphasizes considerably the subjects of groups, theory of numbers, and theory of equations. While the treatment is often, perhaps necessarily, scrappy (by reason of its "synoptic" nature) it has been shown above that the separate sections do not always treat the subject with the breadth to be expected in a volume designed to guide a student to "points from which he can overlook domains of considerable extent." Either the amateur or the professional mathematician must find much of interest within its covers, however, and the work is heartily recommended.

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SHORTER NOTICES.

A History of Elementary Mathematics with Hints on Methods of Teaching. By FLORIAN CAJORI. New York, The Macmillan Company, 1917. viii + 324 pp. Price $1.75.

Two of the most unsatisfactory literary labors that members of our guild are called upon to undertake are first, the publication of a new edition of an old work, using plates of many years' standing; and second, the review of an edition prepared under such circumstances. Probably the most severe critic of the work in hand is Professor Cajori himself, and if he had been free to do so he would undoubtedly have revised the work more radically than most of his readers would or could undertake to do. The book was written more than twenty years ago, and the world knows more about the subject than it did at that time; it has found better ways of overcoming certain difficulties in the matter of presentation of material; and it has the problem of a history of mathematics better in hand.

125 and 272 (add to index) differs from the index form; for Graup, page 161, line 4 from bottom, read Graap; for McMahon, page 293, read MacMahon.

Here and above the slips or errors are practically all of somewhat minor importance. As a whole Professor Miller's work is exceedingly accurate.

Professor Cajori's later contributions to the subject, such as those on Oughtred, the slide rule, and the calculus, are of an entirely different type from those embodied in the book under review. This is due not merely to the fact that they are intended for more mature readers; it is due quite as much to the fact that they are the product of a hand that has acquired new cunning and of a mind that has developed new powers.

It is therefore manifestly unfair to review this book as if it had just been written, and the proper course would seem to be to call attention to the changes actually made in the plates and to a few of the problems which have to be met by all who engage in historical writing and which are suggested by this work. The book has been a guide to many who, in the last two decades, have begun their study of the history of mathematics, and such a record is in itself a happy criticism.

Most of the misprints in the first edition have been corrected, although the elimination of such errors is not complete. Less attention has been paid to the correction of dates, a difficult problem at best. The impression that Ahmes wrote in hieroglyphics (page 23) still remains, as does the one that the subtractive principle of notation originated with the Romans (page 8), and the one that Hypsicles and Ptolemy were contemporaries (page 28), but perhaps slips of this kind should be considered charitably in an introductory work.

The additions of chief importance relate to the Maya zero (page 18), the history of logarithms (pages 164–166), the dollar sign (page 223), and certain modern movements in the teaching of elementary mathematics (pages 287–309). Of these, the second is particularly welcome as answering, at least negatively, the question of the originality of Speidell in the matter of the natural logarithms. While the invention of such logarithms is not definitely placed, a step has been made by Professor Cajori towards doing so. The fourth of the leading additions is also welcome, for it sets forth various features of modern geometry that have developed since the first edition of the work appeared. Certain recent developments in teaching, such as the Perry movement and the work of the International Commission and of various American associations, are also mentioned briefly but with sufficient clearness for the information of teachers.

The work is, therefore, something besides a mere reprint, and will serve a good purpose not only for those who are

beginning their reading in the history of mathematics but also for those who are interested in the teaching of the subject.

As to the general questions suggested by the work, three may profitably be considered: First, how shall Oriental names be transliterated? If we take the Suter list as given in the Abhandlungen zur Geschichte der Mathematik, which is certainly the most scientific one that has been worked out in mathematics, we run into such a difficulty as that of the sound denoted by the English J. In this respect the scheme is not convenient for English, French, and Spanish readers. If this is settled, what shall be the method adopted for Chinese names? Shall Chu be Chou, Tschu, Tscheou, or Chiu, and shall we have the Yih King, the Yi Ching, or the I Ching as the title of a well-known classic? The same difficulty arises with respect to the languages of India, and indeed of all countries not using the Latin letters. In the case of countries where these letters are used, we can easily fix a satisfactory rule, namely, to spell the name as was most commonly done by the individual, as in the case of Widman and Pacioli (not Widmann and Paciuoli), leaving the pronunciation to be determined by the reader's linguistic powers. The question is not an easy one to answer when transliteration is necessary, and Professor Cajori would be among the first to agree that he has not seriously attempted to meet it.

A second question that arises relates to the sequence of topics. Shall a book of this kind be written with a respect to general chronological sequence, or shall it rather consider the question of the sequence of subjects? Do we wish to know what was happening in the thirteenth century, for example, or do we wish to have placed before us the growth of such topics as numerical calculation, algebraic symbolism, and the like? It is no adverse criticism of the work in hand to say that the former plan is generally followed in the first part of the book, and the latter in that part which relates to the last three centuries. For the purposes in view, this is possibly the best plan. The question is, however, a difficult one to answer in planning such a book.

A third question relates to source material. How much material of this kind may properly enter into a book for beginners? The question has been raised of late years by many teachers of history and is a serious one. For example, the hieroglyphics on page 23 are not from Ahmes, but does this

detract from their value? The cubic on page 228 is not given as it is in Cardan's work; but again, is it therefore less valuable to such as will read this book? It is probable that the question cannot be definitely answered as to how much material of this kind should be given, and probably there is a fair basis for argument as to how exactly it should be reproduced.

In view of the fact that no history of mathematics has ever been written that has not been subjected or could not be subjected to severe criticism with respect to its statements, as witness Eneström's perennial criticisms of Cantor, it will hardly be expected that Professor Cajori's latest edition will, under all the circumstances of its publication, be found to be without flaw. It is, however, interesting to see that the work has been found to be so helpful as to warrant this partial revision. Readers may be pardoned if they join with the author in the regret that a radical revision was impossible for reasons that are apparently commercial.

DAVID EUGENE SMITH.

A Text-Book on Practical Mathematics for Advanced Technical Students. By H. LESLIE MANN, B.Sc., A.R.C.Sc. Longmans, Green and Co. (New York), London, 1915. xi + 487 pp. Price $2.10 net.

As is the case with many texts of this type, the material and its arrangement are determined by some restricted need and the personal preference of the author. This text, we are told, is based on the work of the senior students at the Woolwich Polytechnic, following the line of the author's lectures there to the students in mechanical and electrical engineering during the past nine years. Though based on the work of the seniors, "the book is meant to cover a two or three-years' course" assuming "a knowledge of the fundamental principles of algebra, trigonometry, and mensuration, and the use of logarithms and squared paper."

The arrangement will be likely to impress all teachers of mathematics and most teachers of engineering in America as rather reverse in portions of the book to their accustomed order. In several places we were given the sensation of moving backward into the subject. We cannot see what personal preference led to the scrappy, unsystematic treatment of algebra covering seventeen full pages, eight of which are devoted to approximations and applications of approximate

computations, and to giving a cumbersome way of getting the approximate roots of a quadratic equation. Radicals or irrational quantities seem throughout to be replaced by their value to two or three places even when they occur in denominator or divisor. The proper use of significant figures has not received attention.

An elaborate review of trigonometry, including complex quantities and their graphic representation, pages 19–60, is neat and serves its purpose well; yet in passing we observe the order of the cases in the solution of triangles is: I. When three sides are given, II. When two sides and an angle are given, III. When one side and two angles are given. We are far more surprised to find very little of analytic geometry and to have that little introduced in Chapter V by the geometry of three dimensions, pages 61–78, with a touch of plane analytics on the next two pages in defining slope and derivative and no more until Chapters VIII and IX, pages 115–142, which treat also minimum point, point of inflexion, maxima and minima, tangent and normal, angle between a line and a curve, curvature and center of curvature, in order. The treatment of integrations, pages 143–191, and of definite integrals, pages 192–221, is possibly more nearly in the usual order than any other portions of the book. The remainder of the text is devoted to applications of the former subjects. The numerous exercises are very good and seem to come from actual engineering experience.

The last 60 pages are devoted to finite differences and applications. In paragraphs 217 and 220, where the matter of fitting curves to data is treated under the titles "The straight line law" and "General determination of laws," the author seems not to recognize the fact that in curve-fitting the most difficult part by far is the selection of the *type* of equation, but dismisses it by saying "a probable law connecting x and y is known or assumed, and it is necessary to prove that the given tabular values do actually satisfy the law," page 448, and "It is highly probable that one of the three laws

$$y = a + bx^n \quad (1), \quad y = b(x + a)^n \quad (2), \quad y = a + be^{nx} \quad (3)$$

will suit the given values," page 452. It is due the author to state that he has much company in failing to recognize the above fact, and that we are very glad to see these last sixty pages.

We admire the author's aim in making the calculus the center and unity of the book, and his position in claiming that "the calculus cannot be successfully applied to the problems which occur in actual practice until the student has become thoroughly familiar with its underlying principles and methods, and this familiarity can only be obtained by steady practice. It is unfair to a student to give him as a standard form

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

and then expect him to use it as a formula to integrate any function which might resemble it, or by some means reduce to it. This might be working along the line of least resistance, but it is not educational: neither is it to the best interests of the student to whom sound work in differentiation and integration is an absolute necessity."

The text closes with a list of the answers to all the hundreds of exercises and an index. Example 27, page 17, attracted our attention, so we solved it only to find the "true value" .34315 instead of .3429 in the first part, and in the second part .3774 instead of .3781, making the percentage of error 0.318 instead of 0.503 per cent as printed. The difference is probably due to the fact that we rationalized the denominators and divided. The answers should not be carried out so far at any rate.

The publishers have made the book attractive; even the relative dimensions as well as the appearance of the pages are inviting.

CHARLES C. GROVE.

NOTES.

THE April meeting of the Chicago Section of the American Mathematical Society will be devoted in part to a symposium on divergent series and modern theories of summability, the principal speakers being R. D. CARMICHAEL and C. N. MOORE.

AT the annual meeting of the Mathematical Association of America E. V. HUNTINGTON was elected president, D. N.

LEHMER and J. W. YOUNG vice-presidents, and W. D. CAIRNS secretary-treasurer. By amendment of the constitution the office of managing editor was divided, and H. E. SLAUGHT was appointed manager and R. D. CARMICHAEL editor-in-chief of the *Monthly*. The attendance at the meeting numbered 119, including 94 members; 27 persons and institutions were elected to membership. The finances of the association are evidently well managed, the treasurer's report showing a gain of about one hundred dollars for the past year, notwithstanding the cost of the *Monthly*, the subvention paid to the *Annals of Mathematics*, and a small fund set aside for the work of the national committee on mathematical requirements.

THE opening (January) number of volume 19 of the *Transactions of the American Mathematical Society* contains the following papers: "Rational approximations to irrational complex numbers," by L. R. FORD; "On the imaginary roots of a polynomial and the real roots of its derivative," by H. B. MITCHELL; "Relations entre les notions de limite et de distance," by MAURICE FRÉCHET; "On the foundations of the calcul fonctionnel of Fréchet," by A. D. PITCHER and E. W. CHITTENDEN; "Existence theorems for the general real self-adjoint linear system of the second order," by H. J. ETTLINGER; "On boundary value problems in linear differential equations in general analysis," by T. H. HILDEBRANDT; "A fundamental system of formal covariants modulo 2 of the binary cubic," by O. E. GLENN.

THE opening (January) number of volume 40 of the *American Journal of Mathematics* contains: "Flat-sphere geometry. Second paper," by JOHN EIESLAND; "Irrational involutions on algebraic curves," by J. V. DEPORTE; "The set of eight self-associated points in space," by J. R. MUSSELMAN; "Associate minimal surfaces," by J. K. WHITTEMORE; "On integral invariants," by F. W. REED; "Fundamental regions for certain finite groups in S_4," by H. F. PRICE.

THE November number (volume 3, number 11) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A necessary and sufficient condition for the existence of a Stieltjes integral," by G. A. BLISS; "Transformations of applicable conjugate nets of curves on surfaces," by L. P. EISENHART; "On bilinear and n-linear functionals," by C. A. FISCHER; "On the deformation

of an n-cell," by OSWALD VEBLEN; "A theorem on series of orthogonal functions with an application to Sturm-Liouville series," by G. D. BIRKHOFF.

AT the annual meeting of the British mathematical association held on January 10, 1918, the following papers were read: By W. P. MILNE, "Uses and functions of a school mathematical library"; by S. BRODETSKY, "Nomography"; by G. GOODWILL, "Some suggestions for the presentment of mathematics in closer touch with reality"; by T. P. NUNN (presidential address), "Mathematics and individuality."

AT the meeting of the Edinburgh mathematical society on January 11, the following papers were read: By Miss E. PAIRMAN, "On a difference equation due to Stirling"; by E. T. WHITTAKER, "On Bernoulli's and Fürstenau's methods for the solution of equations"; by E. M. HORSBURGH, "An approximate formula for the length of an arc of a suspended rope."

AT the annual meeting of the Paris academy of sciences, held December 10, 1917, the following prizes in pure and applied mathematics were awarded, in addition to those noted in the February number of the BULLETIN: The Lalande prize to Mr. R. JONCKHEERE for his catalogue of double stars; the Valz prize to Dr. A. SCHAUMASSE, of the University of Nice, for the discovery of the comet 1917 b; the Petit d'Ormoy prize to Professor P. DUHEM, of the University of Bordeaux, for his cosmological theories of the earth; the Binoux prize to Professor G. TEIXEIRA, of the University of Porto, for his mathematical works, and honorable mention to Mr. A. BORDEAUX; the Saintour prize to Professor H. LEBESGUE for his contributions to the principles of the calculus; the de Parville prize to Professor C. J. DE LA VALLÉE POUSSIN for his Course of infinitesimal analysis and Lessons on Lebesgue integrals. The Vaillant, Damoiseau, Fourneyron, Pontecoulant, Pierson-Perrin, and Guzman prizes were not awarded.

All the prizes are to be awarded in 1918 and following years under the usual conditions. The following prize questions are proposed: For the Damoiseau prize (2000 francs) to be awarded in 1920: "To improve in some important points the work of Poincaré and Liapounoff on the figures of equilibrium of a rotating fluid mass subject to newtonian attraction. The

Academy calls especial attention to the question of stability and the study of infinitesimal oscillations about a stable figure." For the Bordin prize (3000 francs) to be awarded in 1919: "In the theory of integrals of total differentials of the third kind and of double integrals relative to an algebraic function of two independent variables, there has been proved the existence of certain integers, whose value is difficult to obtain and may depend on the arithmetic nature of the coefficients of the equation of the surface corresponding to the function. The Academy asks a detailed study of these numbers in some important special cases." For the Grand prize (3000 francs) to be awarded in 1920: "To improve the theory of the functions of one variable that can be represented by trigonometric series of several arguments that are linear functions of that variable."

THE following university and college teachers of mathematics have recently entered the national military service:

Professor J. L. COOLIDGE, of Harvard University, has been commissioned major in the ordnance department and sent to Europe as head of a scientific mission. Mr. W. E. EDINGTON, of the University of Illinois, has joined the research division of the Signal Service Bureau at Leavenworth, Kan. Dr. L. R. FORD, of Harvard University, has joined the coast artillery. Dr. L. M. KELLS, of the University of Illinois, has entered the officers' reserve training camp at Battle Creek, Mich. Mr. F. D. POSEY, of Lebec, Cal., has been made second lieutenant in the national army. Mr. J. L. WALSH, of Harvard University, has joined the naval reserve.

PROFESSOR VITO VOLTERRA, of the University of Rome, has been elected a foreign associate of the Paris academy of sciences.

AT the University of Illinois, Dr. J. E. MCATEE, of William Jewell College, and Mr. L. L. STEIMLEY, of the University of Kansas, have been appointed instructors in mathematics. Dr. J. R. MUSSELMAN has resigned his instructorship to enter statistical work in one of the government departments at Washington. Mr. H. D. FRARY has resigned to become director of the wood-testing plant for aeroplanes at the University of Wisconsin. Mr. A. W. LARSEN has resigned to accept an instructorship at the University of Kansas.

PROFESSOR ELLERY W. DAVIS, dean of the school of arts and sciences and head of the department of mathematics of the University of Nebraska, died February 3 at the age of sixty years. Professor Davis was one of the earliest members of the American Mathematical Society, having entered in 1891.

DR. R. A. HARRIS, of the U. S. coast and geodetic survey and well known for his researches in the theory of the tides, died January 17 at the age of fifty-four years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOUTROUX (E.). Natural law in science and philosophy. Authorized translation by F. Rothwell. New York, Macmillan, 1914. 8vo. 218 pp.

"F. R. S." See THOMPSON (S. P.).

HANCOCK (H.). Theory of maxima and minima. Boston, Ginn, 1917. 8vo. 14 + 193 pp.

KENYON (A. M.) and LOVITT (W. V.). Mathematics for collegiate students of agriculture and general science. New York, Macmillan, 1917. 8 + 357 pp. Cloth. $2.00

LOVITT (W. V.). See KENYON (A. M.).

MAURUS (E. J.). An elementary course in differential equations. Boston, Ginn, 1917. 12mo. 8 + 51 pp. $0.72

ROTHWELL (F.). See BOUTROUX (E.).

THOMPSON (S. P.). ("F. R. S."). Calculus made easy: being a very-simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the integral calculus. Second edition, enlarged. London, Macmillan, 1917. 12mo. 12 + 265 pp. 2s.

ZONDADARI (E.). Integrazione grafica e studio delle equazioni differenziali ordinarie del primo ordine coi metodi della geometria descrittiva. Milano, Soc. ed. Dante Alighieri (Roma, tip. Nazionale, Bertero), 1917. 8vo. 9 + 112 pp. L. 3.50

II. ELEMENTARY MATHEMATICS.

BARKER (E. H.). Plane trigonometry with tables. Philadelphia, Blakiston, 1917. 8vo. 8 + 172 pp.

CHADSEY (C. E.) and SMITH (J. H.). Efficiency arithmetic. 3 volumes. Boston, Atkinson, Mentzner and Company, 1917. 12mo. Primary: 7 + 280 pp. Intermediate: 6 + 282 pp. Advanced: 5 + 314 pp. $0.40 + 0.40 + 0.45

FOLDBERG (P. T.). Regnebog for Realskoler. 1. Del. København og Kristiania, Gyldendal, 1917. 8vo. 94 pp.

FRIIS-PETERSEN (F.) og JESSEN (J. W. L.). Opgaver i Algebra for Realklassen og det sproglige Gymnasium. København, Gjellerup, 1917. 8vo. 64 pp.

HARPER (G. A.). See NEWELL (M. J.).

JESSEN (J. W. L.). See FRIIS-PETERSEN (F.).

NEWELL (M. J.) and HARPER (G. A.). Plane and solid geometry. Chicago, Row, Peterson and Company, 1917. 12 + 384 pp.

PLATOU (T. H.). Opgaver i Aritmetik og Algebra for Middelskolen. Kristiania, Cammermeyer, 1917. 8vo. 62 pp.

SMITH (J. H.). See CHADSEY (C. E.).

III. APPLIED MATHEMATICS.

BIANCHI (E.). La determinazione del punto da bordo di aeronavi. Parte 1. (Ministero delle armi e munizioni: direzione generale d'aeronautica, istituto centrale aeronautico.) Roma, tip. Unione ed., 1917. 8vo. 158 pp.

——. See DIFFERENZA.

CONTARINO (F.). See DIFFERENZA.

CRAWLEY (E. S.) and EVANS (H. B.). Analytic geometry. Philadelphia, E. S. Crawley, 1918. 14 + 239 pp. $1.60

DEL BUE (A.). Lezione di meccanica generale: cinematica. Città di Castello, Soc. tip. Leonardo da Vinci, 1917. 72 pp. L. 3.00

DIFFERENZA di longitudine fra Roma (M. Mario) e Napoli (Osserv. di Capadimonte) determinate nei mesi di giugno e luglio del 1909 da E. Bianchi, F. Contarino, V. Nobile, e G. Zappa. Parte 1. (R. Commissione geodetica italiana.) Roma, tip. Nazionale, 1917. 4to. 54 pp.

EVANS (H. B.) See CRAWLEY (E. S.).

FARMER (F. M.). Electrical measurements in practice. New York, McGraw-Hill, 1917. 360 pp. $4.00

GRAY (A.). Principles and practice of electrical engineering. 2d edition. New York, McGraw-Hill, 1917. 431 pp. $3.00

HALSEY (F. A.). Handbook for machine designers, shop men and draftsmen. 2d edition. New York, McGraw-Hill, 1917. 561 pp. $5.00

HILLS (R. W.). Machine drawing. New York, McGraw-Hill, 1917. 92 pp. $1.00

JACOBY (H.). Navigation. New York, Macmillan, 1917. $2.25

KENNELLY (A. E.). The application of hyperbolic functions to electrical engineering problems. 2d edition. New York, McGraw-Hill, 1917. 302 pp. $2.50

——. Artificial electric lines. Their theory, mode of construction and uses. New York, McGraw-Hill, 1917. 348 pp. $4.00

LAWS (F. A.). Electrical measurements. New York, McGraw-Hill, 1917. 719 pp. $5.00

LEUTWILER (O. A.). Elements of machine design. New York, McGraw-Hill, 1917. 607 pp. $4.00

MARCOLONGO (R.). Meccanica razionale. Volume 1: Cinematica-statica. Seconda edizione, riveduta ed ampliata. (Manuali Hoepli, N. 352-353.) Milano, Hoepli, 1917. 24mo. 15 + 323 pp. L. 4.50

NOBILE (V.). See DIFFERENZA.

POORMAN (A. P.). Applied mechanics. New York, McGraw-Hill, 1917. 8vo. 12 + 244 pp. $2.00

SCHLAUCH (W. S.). See WENTWORTH (G.).

SCHUSTER (A.) and SHIPLEY (A. E.). Britain's heritage of science. London, Constable, 1917. 16 + 334 pp.

SECRIST (H.). An introduction to statistical methods. A textbook for college students. A manual for statisticians and business executives. New York, Macmillan, 1917. 22 + 482 pp. $2.00

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